

Two tests for sequential detection of a change-point in a nonlinear model

Gabriela CIUPERCA ¹

Université de Lyon, Université Lyon 1, CNRS, UMR 5208, Institut Camille Jordan, Bat. Braconnier, 43, blvd du 11 novembre 1918, F - 69622 Villeurbanne Cedex, France

Abstract

In this paper, two tests, based on CUSUM of the residuals and least squares estimation, are studied to detect in real time a change-point in a nonlinear model. A first test statistic is proposed by extension of a method already used in the literature but for the linear models. It is tested the null hypothesis, at each sequential observation, that there is no change in the model against a change presence. The asymptotic distribution of the test statistic under the null hypothesis is given and its convergence in probability to infinity is proved when a change occurs. These results will allow to build an asymptotic critical region. Next, in order to decrease the type I error probability, a bootstrapped critical value is proposed and a modified test is studied in a similar way.

Simulation results, using Monte-Carlo technique, for nonlinear models which have numerous applications, investigate the properties of the two statistic tests.

Keywords: sequential detection, change-points, weighted CUSUM, bootstrap, size test, asymptotic behavior.

1. Introduction

Our aim is the construction of a test for detecting a change in a parametric nonlinear model $Y_i = f(X_i; \beta_i) + \varepsilon_i$, $i = 1, \dots, n$. The parameter β will be first estimated by a parametric method and hypothesis test will be afterwards made by two nonparametric statistics. The test statistics we are going to consider are based on sequential empirical processes of parametrically estimated residuals. This problem appears in various fields, especially biology (for example: growth model or compartmental model), chemistry, industry (quality control), finance, ...

Generally, there are two types of change-point problem: *a posteriori* and *a priori*(sequential). The *a posteriori* change-point problem arises when the data are completely known at the end of the experiment to process. For this model we begins by finding the change-points number; after that their locations and the regression parameters on each interval are estimated. In the case of a parametric *a posteriori* model with

¹*email:* Gabriela.Ciuperca@univ-lyon1.fr

change-points we can give the following references: for a constant model with K change-points, a consistent estimator for K was proposed by Yao and Au (1988), using the least squares estimation method. If the errors are strongly mixing or long-range-dependent processes, always for a constant model, Lavielle and Moulines (2000) estimate the change-point number using a penalized least-squares approach. Bai (1999) proposes a test based on the likelihood for a linear model. Again, concerning the detection of a change in a linear model we can remind papers based on information criterion of Osorio and Galea (2005), Wu (2008) or still Nosek (2010). In a linear model, but with long memory errors, Belkhouja and Boutahar (2009) use several methods to detect the break number: three information criteria, a sequential parametric test and a procedure based on sum of squared residuals. A large class of time series with change-points are estimated by a semi-parametric framework, but for a known change number, by Bardet et al. (2012). For a parametric nonlinear model, with multiple change-points, a general criterion is proposed by Ciuperca (2011). For the detection of the change-point number by hypothesis test in a linear a posteriori model, we can remind the paper of Liu et al. (2008), where the empirical likelihood test was considered in the particular case to detect a single change in a linear model. Qu and Perron (2007) propose likelihood ratio type statistics to test the null hypothesis K changes, against the alternative hypothesis of $K + 1$ changes, always for a linear model. In the sequential change-point problem, which will be presented here, the detection is performed in real time. In a linear model, the most used technique is the CUSUM method. Horváth et al. (2004) propose two schemes to detect a change in a linear model, results which are improved, using the bootstrapping, by Hušková and Kirch (2012). The same method we find in Xia et al. (2009) for a generalized linear model. In the sequential change-point detection literature most researches consider the detection of a change in the random variable distribution (see e.g. Lai and Xing, 2010, or Mei, 2006). We can also recall several testing procedures proposed by Neumeyer and Van Keilegom (2009) for detecting the change-points in the error distribution of non-parametric regression models.

In this paper, the real time change-point detection in a nonlinear model is studied. Generalizing Horvath et al. (2004) framework, a first test statistic is studied using the weighted CUSUM method, calculated after that the model parameters have been estimated by least squares method. Next, in order to decrease the type I error probability, following the idea introduced by Hušková and Kirch (2012) for the linear case, a modified test (of the first) by bootstrapping is considered. It is important to note that, the nonlinearity changes the results and the approach made by Horvath et al. (2004) and by Hušková and Kirch (2012) for the linear case. Above all, in a linear model, the least squares estimator of the parameters has an explicit expression, which facilitates the calculations and the results proofs. All results proofs are based on the explicit form of the estimator. In the nonlinear case, since the estimator expression is unknown and the regression function derivatives with respect to regression parameters depends on parameters and on regressors as well, imply that the theoretical results (and their proofs) are different. These problems are even more difficult to solve in a model where change-point occurs. Numerical algorithms will also change to calculate the critical value and test the break presence. On the other hand, in the paper of Hušková and Kirch (2012), the fact that the linear model contains intercept (see the Assumption $\mathcal{A}.1(ii)$), influences in an important way the results. It is worth mentioning that we don't impose a discontinuity condition in the change-point for the model. By simulations, for two nonlinear models which have numerous practical applications, we obtain that the two

proposed tests have the empirical power equal to 1 and the empirical sizes widely smaller than the fixed theoretical size. However, the precision of the change-point estimator is the same by both methods.

The paper is organized as follows. In Section 2, we introduce the model assumptions and some general notations. The construction of a statistical test and its asymptotic behavior are presented in Section 3. To decrease the type I error probability, Section 4 presents a modified test by bootstrapping. Next, simulation results illustrate the obtained theoretical results in Section 5. The proofs of the main results are given in Section 6, followed in Appendix by some Lemmas.

2. Model and notations

For coherence, we try to use the some notations as in Hušková and Kirch's paper, where the linear model was considered.

Let us consider the following random parametric nonlinear model with independent observations

$$Y_i = f(\mathbf{X}_i; \boldsymbol{\beta}_i) + \varepsilon_i, \quad i = 1, \dots, m, \dots, m + T_m.$$

For the observation i , Y_i denotes the response variable, \mathbf{X}_i is a $p \times 1$ random vector of regressors, the function $f : \mathbf{R}^p \times \Theta \rightarrow \mathbf{R}$ is known up to the parameters $\boldsymbol{\beta}_i$ of dimension $q \times 1$, $\boldsymbol{\beta}_i \in \Theta \subseteq \mathbf{R}^q$, with Θ a compact set. For the function f we make the classical suppositions for a nonlinear model: $\mathbf{f}(\mathbf{x}; \boldsymbol{\beta})$ is continuous in \mathbf{x} and of class $C^2(\Theta)$. For the function $f(\mathbf{x}; \boldsymbol{\beta})$, we denote $\dot{\mathbf{f}}(\mathbf{x}; \boldsymbol{\beta}) \equiv \partial f(\mathbf{x}; \boldsymbol{\beta}) / \partial \boldsymbol{\beta}$ and $\ddot{\mathbf{f}}(\mathbf{x}; \boldsymbol{\beta}) \equiv \partial^2 f(\mathbf{x}; \boldsymbol{\beta}) / \partial \boldsymbol{\beta}^2$. We suppose that on the first m observations, no change in the parameter regression has occurred

$$\boldsymbol{\beta}_i = \boldsymbol{\beta}^0, \quad \text{for } i = 1, \dots, m,$$

with $\boldsymbol{\beta}^0$ the true value of the parameter on the observations $1, \dots, m$. The value of $\boldsymbol{\beta}^0$ is unknown. We test the null hypothesis, that for all the following observations, there is no change in the model

$$H_0 : \boldsymbol{\beta}_i = \boldsymbol{\beta}^0, \quad m + 1 \leq i \leq m + T_m, \quad (1)$$

against the hypothesis that there is a change to the $m + k_m^0 + 1$ observation

$$H_1 : \exists k_m^0 \geq 1, \text{ such that } \begin{cases} \boldsymbol{\beta}_{i,m} = \boldsymbol{\beta}^0 & \text{for } m + 1 \leq i \leq m + k_m^0 \\ \boldsymbol{\beta}_{i,m} = \boldsymbol{\beta}_m^0 \neq \boldsymbol{\beta}^0 & \text{for } m + k_m^0 + 1 \leq i \leq m + T_m. \end{cases} \quad (2)$$

The value of $\boldsymbol{\beta}_m^0$ is also unknown. This problem has been addressed in the literature if function f is linear $f(\mathbf{x}; \boldsymbol{\beta}) = \mathbf{x}^t \boldsymbol{\beta}$ (see Horváth et al., 2004, Hušková and Kirch, 2012). Let be the sequential detector statistic, built as the weighted cumulative sum of the residuals, for $0 \leq \gamma < 1/2$, $k = 1, \dots, T_m$

$$\begin{cases} \Gamma(m, k, \gamma) \equiv \sum_{m+1 \leq i \leq m+k} \hat{\varepsilon}_i / g(m, k, \gamma) = \sum_{m+1 \leq i \leq m+k} [Y_i - f(\mathbf{X}_i, \hat{\boldsymbol{\beta}}_m)] / g(m, k, \gamma) \\ \text{with } g(m, k, \gamma) \equiv m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{k+m}\right)^\gamma, \end{cases} \quad (3)$$

where $\hat{\boldsymbol{\beta}}_m \equiv \arg \min_{\boldsymbol{\beta}} \sum_{j=1}^m [Y_j - f(\mathbf{X}_j; \boldsymbol{\beta})]^2$ is the least squares (LS) estimator of $\boldsymbol{\beta}$ calculated on the observations $1, \dots, m$. With this estimator we calculate the parametric residuals $\hat{\varepsilon}_i \equiv Y_i - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m)$, for $i = 1, \dots, k$. Recall that the cumulative sum (CUSUM) of the residuals is $\sum_{i=m+1}^{m+k} \hat{\varepsilon}_i$. Let be the $q \times q$ -matrix $\mathbf{B}_m \equiv m^{-1} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0)$ which is supposed non-regular for all m with probability one. Classic asymptotic results for a nonlinear regression (see also the relation (35)) imply $\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0 = \mathbf{B}_m^{-1} \left[m^{-1} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \varepsilon_i \right] (1 + o_P(1))$. The function $g(m, k, \gamma)$ of the relation (3), proposed by Horváth et al. (2004), is used as a boundary. Let us also consider the notations: $\mathbf{A} \equiv E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)]$, $\mathbf{B} \equiv E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}; \boldsymbol{\beta}^0)]$, $A_i \equiv \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) \mathbf{B}^{-1} \mathbf{A}$, $\mathcal{D} \equiv [\mathbf{A}^t \mathbf{B}^{-1} \mathbf{A}]^{1/2}$, $D_A \equiv \mathbf{A}^t \mathbf{A}$. Matrix \mathbf{B} is supposed positive definite. All throughout the paper, vectors and matrices are written in bold face.

The regression function, the random vector \mathbf{X}_i and the error ε_i satisfy the following assumptions:

- (A1) $(\varepsilon_i)_{1 \leq i \leq n}$ are i.i.d. and $E[\varepsilon_i] = 0$, $\text{Var}[\varepsilon_i] = \sigma^2$ and $E[|\varepsilon_i|^\nu] < \infty$ for some $\nu > 2$.
- (A2) $\ddot{f}(\mathbf{x}, \boldsymbol{\beta})$ is bounded for all $\boldsymbol{\beta}$ in a neighborhood of $\boldsymbol{\beta}^0$, for all $\mathbf{x} \in R^p$.
- (A3) For every $i = 1, \dots, T_m$, the errors ε_i are independent of the random vectors \mathbf{X}_j , for all $j = 1, \dots, m + T_m$.
- (A4) $(m + l)^{-1} \sum_{i=1}^{m+l} f(\mathbf{X}_i; \boldsymbol{\beta}^0) \xrightarrow[m \rightarrow \infty]{a.s.} E[f(\mathbf{X}; \boldsymbol{\beta}^0)]$, $(m + l)^{-1} \sum_{i=1}^{m+l} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \xrightarrow[m \rightarrow \infty]{a.s.} E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)]$,
 $(m + l)^{-1} \sum_{i=1}^{m+l} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) \xrightarrow[m \rightarrow \infty]{a.s.} \mathbf{B}$ for all $l = 0, 1, \dots, T_m$.

Assumptions (A2) and (A4) are made for the true parameter $\boldsymbol{\beta}^0$, under null hypothesis H_0 . For the parameter $\boldsymbol{\beta}_m^0$, under the alternative hypothesis, we request only the similar of (A4):

- (A5) $(m + k_m^0 + l)^{-1} \sum_{i=1}^{m+k_m^0+l} f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) \xrightarrow[m \rightarrow \infty]{a.s.} E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)]$, $(m + k_m^0 + l)^{-1} \sum_{i=1}^{m+k_m^0+l} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}_m^0) \xrightarrow[m \rightarrow \infty]{a.s.} E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}_m^0)]$,
 $(m + k_m^0 + l)^{-1} \sum_{i=1}^{m+k_m^0+l} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}_m^0) \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}_m^0) \xrightarrow[m \rightarrow \infty]{a.s.} \mathbf{B}$, for all $l = 0, 1, \dots, T_m - k_m^0$.

The assumption that the nonlinear function f is continuous in \mathbf{x} , of class C^2 in $\boldsymbol{\beta}$ and also assumptions (A2) and (A4) are commonly used in nonlinear modeling and are necessary for the consistency and the asymptotic normality of the LS parameter estimator (see e.g. Seber and Wild, 2003). Furthermore, the two values $\boldsymbol{\beta}^0$ and $\boldsymbol{\beta}_m^0$ are interior points of the set Θ .

The error variance σ^2 is unknown. To estimate it, on the historical observations $i = 1, \dots, m$, we consider an consistent estimator

$$\hat{\sigma}_m^2 \equiv \frac{1}{m - q} \sum_{j=1}^m [Y_j - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_m)]^2. \quad (4)$$

For the errors, let us consider: $\bar{\varepsilon}_{m+k} = (m + k)^{-1} \sum_{j=1}^{m+k} \varepsilon_j$, $\overline{\varepsilon}_{m+k}^2 = (m + k)^{-1} \sum_{j=1}^{m+k} \varepsilon_j^2$, and then, an another estimator for its variance besides of (4), built on the $m + k$ first observations, is $\hat{\sigma}_{m,k}^2 \equiv (m + k)^{-1} \sum_{i=1}^{m+k} (\varepsilon_i - \bar{\varepsilon}_{m+k})^2$.

Two cases are possible for the sample size, which will give different results, under the null hypothesis for

the test statistics:

- $T_m = \infty$, the open-end procedure;
- $T_m < \infty$, $\lim_{m \rightarrow \infty} T_m = \infty$, with $\lim_{m \rightarrow \infty} \frac{T_m}{m} = T > 0$, with the possibility $T = \infty$. In this case we have the closed-end procedure.

Concerning the used norms, for a p -vector $\mathbf{v} = (v_1, \dots, v_p)$, let us denote by $\|\mathbf{v}\|_1 = \sum_{j=1}^p |v_j|$ its L^1 -norm and $\|\mathbf{v}\|_2 = (\sum_{j=1}^p v_j^2)^{1/2}$ its L^2 -norm. For a matrix $\mathcal{M} = (a_{ij})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}}$, we denote by $\|\mathcal{M}\|_1 = \max_{j=1, \dots, q} (\sum_{i=1}^p |a_{ij}|)$ the subordinate norm to the vector norm $\|\cdot\|_1$ and by $\|\mathcal{M}\|_2 = \sqrt{\rho(\mathcal{M}\mathcal{M}^t)}$ the subordinate norm to $\|\cdot\|_2$, with $\rho(\mathcal{M}\mathcal{M}^t)$ the spectral radius of $\mathcal{M}\mathcal{M}^t$.

All throughout the paper, C denotes a positive generic constant which may take different values in different formula or even in different parts of the same formula. All vector are column and \mathbf{v}^t denotes the transpose of \mathbf{v} . We say that a random variable set (V_n) is bounded by a constant C with a probability close to 1 (or with a probability arbitrarily large): $\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N}$ such that $\mathbf{P}[V_n > C] < 1 - \epsilon$.

Now, a notation and a relation on the function g , used for the result proofs. Using the relation that for all $x > 0$ we have $0 < \frac{x}{1+x} < 1$ and that $\gamma \in [0, 1/2)$, we obtain that

$$K_m \equiv \sup_{1 \leq k < \infty} \frac{km^{-1/2}}{g(m, k, \gamma)} = \sup_{1 \leq k < \infty} \left(\frac{k/m}{1 + k/m} \right)^{1-\gamma} \in [0, 1]. \quad (5)$$

After from these general notations, in every section we shall give the notations used for each test.

The proofs of all main results of Sections 3 and 4 are given in Section 6. To prove these results, necessary lemmas are stated and proved in Appendix (Section 7).

3. Test by weighted CUSUM, without bootstrapping

We are going first to build a test statistic based on the residuals $\hat{\epsilon}_i = Y_i - f(\mathbf{X}_i; \hat{\beta}_m)$ after the observation m by estimating the parameter β on the historical data $(Y_i, \mathbf{X}_i)_{1 \leq i \leq m}$. The study of this statistic will be hampered by the fact that the estimator $\hat{\beta}_m$ does not have an explicit expression.

The following Theorem is the generalization of the result obtained by Horváth et al.(2004) for the linear model, on the asymptotic distribution of the test statistic under the null hypothesis given by (1). We remark that, unlike to the linear case, the asymptotic distribution of the test statistic, under H_0 , depends on the function $f(\mathbf{x}; \beta^0)$ and on the true parameter β^0 . The value of T_m , with respect to m , also influence the asymptotic distribution.

Theorem 3.1 *Let us consider the assumptions (A1)-(A4). Under the null hypothesis H_0 specified by (1), for all real $c >$, we have*

(i) If $T_m = \infty$ or ($T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = \infty$), then

$$\lim_{m \rightarrow \infty} \mathbf{P} \left[\frac{1}{\hat{\sigma}_m} \sup_{1 \leq k < \infty} \left| \sum_{i=m+1}^{m+k} \hat{\varepsilon}_i \right| / g(m, k, \gamma) \leq c \right] = \mathbf{P} \left[\sup_{0 \leq t \leq \frac{1}{\mathcal{D}^2}} \frac{(1+t-\mathcal{D}^2 t)|W(t)|}{t^\gamma} \leq c \right]. \quad (6)$$

(ii) If $T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = T < \infty$, then the left-hand side of (6) is equal to $\mathbf{P} \left[\sup_{0 \leq t \leq \frac{T}{1+\mathcal{D}^2 T}} \frac{(1+t-\mathcal{D}^2 t)|W(t)|}{t^\gamma} \leq c \right]$. Here $\{W(t), 0 \leq t < \infty\}$ is a Wiener process (Brownian motion) i.e. a centered Gaussian process, with covariance function $\text{Cov}(W(s), W(t)) = \min(s, t)$, $s, t \in [0, \frac{1}{\mathcal{D}^2}]$ for (i) and $s, t \in [0, \frac{T}{1+\mathcal{D}^2 T}]$ for (ii).

In order to have a test statistic, thus, to build a critical region, it is necessary to study the behavior of the statistic in the left-hand side of (6) under the alternative hypothesis H_1 . By the following Theorem, this statistic converges in probability to infinity as $m \rightarrow \infty$. For this, we suppose that the change-point k_m^0 is not very far from the last observation of historical data. Obviously, this supposition poses no problem for practical applications, since if hypothesis H_0 was not rejected until an observation k_m of order m , we reconsider as historical data, all observations of 1 to k_m . Another supposition is that, before and after the break, on average, the model is different, without imposing a discontinuity condition in the change-point.

Theorem 3.2 Suppose that the assumptions (A1)-(A5) hold. Under the alternative hypothesis H_1 specified by (2), if $k_m^0 = O(m)$ and $E[f(\mathbf{X}; \boldsymbol{\beta}^0)] \neq E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)]$ hold also, then

$$\frac{1}{\hat{\sigma}_m} \sup_{1 \leq k \leq T_m} \left[\left| \sum_{i=m+1}^{m+k} \hat{\varepsilon}_i \right| / g(m, k, \gamma) \right] \xrightarrow{P}_{m \rightarrow \infty} \infty.$$

Considering the Theorems 3.1 and 3.2 we derive in the next corollary a test statistic for testing the lack of change against the break presence.

Corollary 3.1 Consequence of these two theorems, following statistic can be used to test H_0 against H_1 :

$$Z_\gamma(m) \equiv \frac{1}{\hat{\sigma}_m} \sup_{1 \leq k \leq T_m} \left| \sum_{i=m+1}^{m+k} \hat{\varepsilon}_i \right| / g(m, k, \gamma). \quad (7)$$

The asymptotic critical region is $\{Z_\gamma(m) \geq c_\alpha(\gamma)\}$, where $c_\alpha(\gamma)$ is the $(1 - \alpha)$ quantile of the distribution of $\sup_{0 \leq t \leq \frac{1}{\mathcal{D}^2}} [t^{-\gamma}(1+t-\mathcal{D}^2 t)|W(t)|]$, if $\lim_{m \rightarrow \infty} T_m/m = \infty$, and of $\sup_{0 \leq t \leq \frac{T}{1+\mathcal{D}^2 T}} [t^{-\gamma}(1+t-\mathcal{D}^2 t)|W(t)|]$, if $\lim_{m \rightarrow \infty} T_m/m = T \in (0, \infty)$. For some given $\alpha \in (0, 1)$, this statistical test, consequence of Theorems 3.1 and 3.2, has the asymptotic type I error probability (size) α and the asymptotic power 1.

It is important to note that, in the linear case $f(\mathbf{x}; \boldsymbol{\beta}) = \mathbf{x}'\boldsymbol{\beta}$, the value of \mathcal{D} depends only on $E[\mathbf{X}]$, $E[\mathbf{X}\mathbf{X}']$ but not on the values of $\boldsymbol{\beta}^0$. For a nonlinear model, the critical values $c_\alpha(\gamma)$ depend on the regression function f , the distribution of random vector \mathbf{X} and on parameter value $\boldsymbol{\beta}^0$ before the change-point.

Remark 1 In the linear case, the assumption that the model contains intercept, $\mathbf{X} = (1, X_1, \dots, X_p)$, $\boldsymbol{\beta} = (b_0, b_1, \dots, b_p)$, imposed by Horváth et al. (2004), is essential. If $E[X_1] = \dots = E[X_p] = 0$, then it is necessary that the model has different intercepts before and after change-point. Without this supposition, the test statistic $Z_\gamma(m)$ can not converge to infinity under H_1 .

Therefore, we deduce from it that, the null hypothesis H_0 is rejected in the change-point

$$\hat{\tau}_m \equiv \begin{cases} \inf \left\{ 1 \leq k \leq T_m, \hat{\sigma}_m^{-1} |\Gamma(m, k, \gamma)| \geq c_\alpha(\gamma) \right\} \\ \infty, \text{ if } \hat{\sigma}_m^{-1} |\Gamma(m, k, \gamma)| < c_\alpha(\gamma), \text{ for every } 1 \leq k \leq T_m. \end{cases} \quad (8)$$

which we can consider as estimator for k_m^0 .

4. Test by weighted CUSUM, with bootstrapping

In order to improve the critical values of the test, thus, to decrease the type I error probability, we extend the method proposed by Hušková and Kirch (2012), which uses the bootstrapping to calculate the critical value, function of the observation position, after the observation m .

Let us suppose that until the observation $m + k$, the hypothesis H_0 has not been rejected yet. Thus, for $l = 1, \dots, m + k$ we have that under H_0 , using the relation (35) and the proof of the Lemma 7.1, the cumulative sum of the residuals defined by (3) can be approached

$$\Gamma(m, l, \gamma) = \left[\sum_{i=m+1}^{m+l} \varepsilon_i - \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_j \right) \mathbf{B}_m^{-1} \sum_{i=m+1}^{m+l} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \right] / g(m, l, \gamma) (1 + o_P(1)).$$

In order to realize the bootstrapping, let us consider the discrete uniform random variables $\mathcal{U}_{m,k}(i)$, for $i = 1, \dots, m + T_m$, such that $P[\mathcal{U}_{m,k}(i) = j] = 1/(m + k)$, for $j = 1, \dots, m + k$. We denote also by $\mathbf{P}_{m,k}^*$, $\mathbf{E}_{m,k}^*$, $\text{Var}_{m,k}^*$ the conditional probability, expectation, variance we respect to $\{\mathcal{U}_{m,k}(i), 1 \leq i \leq m + T_m\}$, given $(Y_j, \mathbf{X}_j)_{1 \leq j \leq m+k}$. The conditional expectation with the bootstrapped regressors is, for $i = 1, \dots, m + T_m$,

$$\mathbf{E}_{m,k}^*[\dot{\mathbf{f}}(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0)] = \frac{1}{m + k} \sum_{j=1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0).$$

Keeping the same notations as in the linear model of Hušková and Kirch (2012), let us consider (see Section 2, for the other notations), for $k = 1, \dots, T_m$, following notations

- $\mathbf{c}_1(m, k, l) \equiv D_A^{-1} \mathbf{B}_m \left[\sum_{i=m+1}^{m+l} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \mathbb{1}_{l \leq k} + \sum_{i=m+k-l+1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \mathbb{1}_{k < l < m+k} + l(m + k)^{-1} \sum_{i=1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \mathbb{1}_{l \geq m+k} \right]$, for $1 \leq l \leq T_m$. In the linear model, $\mathbf{c}_1(m, k, l)$ depends only \mathbf{X}_i .
- $\tilde{\Gamma}(m, k, l, \gamma)(\varepsilon_1, \dots, \varepsilon_{m+l}) \equiv \left[\sum_{i=m+1}^{m+l} \varepsilon_i - \left(m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_j \right) \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l) \right] / g(m, l, \gamma)$, which is an approach of the weighted CUSUM statistic $\Gamma(m, l, \gamma)$ given by (3), in order to facilitate the bootstrap.

- $\hat{\varepsilon}_{m,k}(j) \equiv Y_j - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})$ are the residuals from the ordinary least squares method, with $\hat{\boldsymbol{\beta}}_{m+k} \equiv \arg \min_{\boldsymbol{\beta}} \sum_{j=1}^{m+k} [Y_j - f(\mathbf{X}_j; \boldsymbol{\beta})]$.
- $\varepsilon_{m,k}^*(i) \equiv \hat{\varepsilon}_{m,k}(\mathcal{U}_{m,k}(i))$ are the bootstrap errors.
- $\hat{\sigma}_{m,k}^{(*)2} \equiv (m-q)^{-1} \sum_{i=1}^m \left[\varepsilon_{m,k}^*(i) - \left(m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{m,k}^*(j) \right) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \right]^2$ the bootstrap variance estimator.
- $F_{m,k}^*(x) \equiv \mathbf{P}_{m,k}^* \left[1/\hat{\sigma}_{m,k}^{(*)} \sup_{1 \leq l \leq T_m} |\tilde{\Gamma}(m, k, l, \gamma)(\varepsilon_{m,k}^*(1), \dots, \varepsilon_{m,k}^*(m+l))| \leq x \right]$ a distribution function calculated using the bootstrap results.
- For $N \geq 1$, let us consider $\tilde{F}_{m,k} \equiv \sum_{i=0}^{N-1} \alpha_i F_{m, \max((j-i)L, 0)}^*$, for $k = jL, \dots, (j+1)L - 1$ an other distribution function, proposed by Hušková and Kirch (2012) in order to accelerate the procedure. The positive constants α_i are such that $\sum_{i=0}^{N-1} \alpha_i = 1$.

We note that in order to calculate the bootstrapped residuals $\varepsilon_{m,k}^*(i)$, only the data $(Y_i, \mathbf{X}_i)_{1 \leq i \leq m+T_m}$ are bootstrapped, not the estimator $\hat{\boldsymbol{\beta}}_{m+k}$ of $\boldsymbol{\beta}$ calculated on not bootstrapped data. The $(1 - \alpha)$ quantile $c_{m,k;\alpha}(\gamma)$ at time $m+k$ of the distribution $\tilde{F}_{m,k}$ is obtained as the smallest real value such that

$$\tilde{F}_{m,k}(c_{m,k;\alpha}(\gamma)) \geq 1 - \alpha. \quad (9)$$

Contrary to the case of Corollary 3.1, for the weighted CUSUM statistic without bootstrapping, the critical values $c_{m,k;\alpha}(\gamma)$ depend at the same time of m , and k besides α and γ .

Before to state the main results of this section, let us recall the Hájek-Rényi inequality (see Hájek and Rényi, 1955) that is a generalization of the Kolmogorov inequality.

Hájek-Rényi inequality: if $(G_k)_{1 \leq k \leq n}$ is a sequence of independent random variables with $E[G_k] = 0$, $\text{Var}(G_k) < \infty$ and $(b_k)_{1 \leq k \leq n}$ is a non-decreasing sequence of positive numbers, then, for any $\epsilon > 0$ and $m \leq n$,

$$\mathbf{P} \left[\max_{m \leq k \leq n} \left| \frac{\sum_{j=1}^k G_j}{b_k} \right| \geq \epsilon \right] \leq \frac{1}{\epsilon^2} \left[\sum_{j=m+1}^n \frac{E[G_j^2]}{b_j^2} + \sum_{j=1}^m \frac{E[G_j^2]}{b_m^2} \right].$$

A particular case of this inequality is we consider $b_k = g(n, k, \gamma)$, which is an increasing sequence in k , with the function g specified by relation (3).

For the linear model (see Hušková and Kirch, 2012), to study the behavior of the distribution function $\tilde{F}_{m,k}$, then the behavior of the statistic $1/\hat{\sigma}_{m,k}^{(*)} \sup_{1 \leq l \leq T_m} |\tilde{\Gamma}(m, k, l, \gamma)(\varepsilon_{m,k}^*(1), \dots, \varepsilon_{m,k}^*(m+l))|$, the Hájek-Rényi inequality alone was sufficient. In the nonlinear model, in the calculation of the bootstrapped residual $\varepsilon_{m,k}^*$, then of $\hat{\varepsilon}_{m,k}$, the LS estimator $\hat{\boldsymbol{\beta}}_{m+k}$ intervenes. Since $\hat{\boldsymbol{\beta}}_{m+k}$ was not an explicit expression, we need a generalization of this inequality for random variable sequence of expectation converging uniformly to 0. First, we have the following general result.

Proposition 4.1 *If $(Z_{k,n})_{1 \leq k \leq n}$ is a random variable such that $E[Z_{k,n}] = \mu_{k,n} \rightarrow 0$, for $n \rightarrow \infty$, uniformly in k , and for all $\epsilon > 0$ and $m \leq n$, $P[\max_{m \leq k \leq n} |Z_{k,n} - \mu_{k,n}| \geq \epsilon] \rightarrow 0$, then, there exists a natural number n_ϵ such that for $n \geq n_\epsilon$, $P[\max_{m \leq k \leq n} |Z_{k,n}| \geq 2\epsilon] \rightarrow 0$.*

As a consequence of the Proposition 4.1 and of the Hájek-Rényi inequality, a generalization of this last one can be established, for random variables with the expectation converging to 0. Let $(G_j)_{1 \leq j \leq n}$ be a sequence of random variables such that $E[G_j^2] < \infty$, for all $j = 1, \dots, n$ and $E[b_k^{-1} \sum_{j=1}^k G_j] = \mu_{k,n} \rightarrow 0$, uniformly in k , for $n \rightarrow \infty$, with the positive sequence $(b_k)_{1 \leq k \leq n}$ non-decreasing. Then, by the proof of Proposition 4.1, we have that, for any $\epsilon > 0$, there exists a natural number n_ϵ such that for $n \geq n_\epsilon$

$$P \left[\max_{1 \leq k \leq n} \frac{|\sum_{j=1}^k G_j|}{b_k} \geq 2\epsilon \right] \leq P \left[\max_{1 \leq k \leq n} \frac{|\sum_{j=1}^k (G_j - E[G_j])|}{b_k} \geq \epsilon \right]. \quad (10)$$

On the other hand, by the Hájek-Rényi inequality, we have for the random variable $G_j - E[G_j]$, for any $\epsilon > 0$,

$$P \left[\max_{1 \leq k \leq n} \frac{|\sum_{j=1}^k (G_j - E[G_j])|}{b_k} \geq \epsilon \right] \leq \frac{1}{\epsilon^2} \sum_{j=1}^n \frac{\text{Var}[G_j]}{b_j^2}. \quad (11)$$

But $\text{Var}[G_j] \leq E[G_j^2]$. By the relations (10) and (11) it follows immediately that, for any sequence of random variables $(G_j)_{1 \leq j \leq n}$ such that $E[G_j^2] < \infty$, for all $j = 1, \dots, n$ and $E[b_k^{-1} \sum_{j=1}^k G_j] = \mu_{k,n} \rightarrow 0$, uniformly in k , for $n \rightarrow \infty$, with the positive sequence $(b_k)_{1 \leq k \leq n}$ non-decreasing and for any $\epsilon > 0$, then, there exists a natural number n_ϵ such that for $n \geq n_\epsilon$,

$$P \left[\max_{1 \leq k \leq n} \frac{|\sum_{j=1}^k G_j|}{b_k} \geq 2\epsilon \right] \leq \frac{1}{\epsilon^2} \sum_{j=1}^n \frac{E[G_j^2]}{b_j^2}. \quad (12)$$

Now, in order to study the residuals $\hat{\varepsilon}_{m,k}(i) = Y_i - f(\mathbf{X}_i; \hat{\beta}_{m+k})$, calculated after observation m , we underline, by a decomposition, the corresponding model error ε_i . Depending on the position of the observation "i" with respect to change-point $m + k_m^0$, where k_m^0 is the change-point position under the alternative hypothesis H_1 given by (2), and on the position of k with respect to k_m^0 , we have the decomposition for the residuals

$$\hat{\varepsilon}_{m,k}(i) = \varepsilon_i + f(\mathbf{X}_i; \beta^0) \mathbb{1}_{i \leq m+k_m^0} + f(\mathbf{X}_i; \beta_m^0) \mathbb{1}_{i > m+k_m^0} - f(\mathbf{X}_i; \hat{\beta}_{m+k}) \mathbb{1}_{i \leq m+k_m^0} [\mathbb{1}_{k \leq k_m^0} + \mathbb{1}_{k > k_m^0} - f(\mathbf{X}_i; \hat{\beta}_{m+k}) \mathbb{1}_{i > m+k_m^0} \mathbb{1}_{k > k_m^0}].$$

Since $\hat{\beta}_{m+k}$ is the least squares estimator of β , we have $0 = \sum_{i=1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_{m+k}) [\varepsilon_i - f(\mathbf{X}_i; \hat{\beta}_{m+k}) + f(\mathbf{X}_i; \beta^0)] \mathbb{1}_{k \leq k_m^0} + \sum_{i=m+k_m^0+1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_{m+k}) [\varepsilon_i - f(\mathbf{X}_i; \hat{\beta}_{m+k}) + f(\mathbf{X}_i; \beta_m^0)] \mathbb{1}_{k > k_m^0}$. Then $\sum_{i=1}^{m+k} \varepsilon_i \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_{m+k}) = \sum_{i=1}^{m+k} f(\mathbf{X}_i; \hat{\beta}_{m+k}) \cdot \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_{m+k}) - \sum_{i=1}^{m+k_m^0} f(\mathbf{X}_i; \beta^0) \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_{m+k}) - \sum_{i=m+k_m^0+1}^{m+\min(k, k_m^0)} f(\mathbf{X}_i; \beta_m^0) \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_{m+k}) \mathbb{1}_{k > k_m^0}$. The statistic $g(m, l, \gamma) \tilde{\Gamma}(m, l, \gamma) (\varepsilon_{m,k}^*(1), \dots, \varepsilon_{m,k}^*(m+l))$ becomes

$$\sum_{i=m+1}^{m+l} \hat{\varepsilon}_{m,k}(\mathcal{U}_{m,k}(i)) - \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \beta^0) \hat{\varepsilon}_{m,k}(\mathcal{U}_{m,k}(j)) \right) \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l) \equiv I_1 + I_2 + \mathcal{R}_m, \quad (13)$$

with $I_1 \equiv \sum_{i=m+1}^{m+l} \varepsilon_{\mathcal{U}_{m,k}(i)}$ and $I_2 \equiv -\left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{\mathcal{U}_{m,k}(j)}\right) \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l)$. The expression of \mathcal{R}_m will be specified in Appendix (Section 7). We precise that the bootstrapped residuals are $\hat{\varepsilon}_{m,k}(\mathcal{U}_{m,k}(i)) = Y_{\mathcal{U}_{m,k}(i)} - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k})$ and $\varepsilon_{\mathcal{U}_{m,k}(i)} = Y_{\mathcal{U}_{m,k}(i)} - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0) \mathbb{1}_{\mathcal{U}_{m,k}(i) \leq m+k_m^0} - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}_m^0) \mathbb{1}_{\mathcal{U}_{m,k}(i) > m+k_m^0}$.

With these elements, we can prove that the statistic $\tilde{\Gamma}(m, k, l, \gamma)$ is asymptotically determined by I_1 and I_2 under H_0 and that each of them converges to a Wiener process. For these, we prove, by the following Proposition, that the term I_2 can be also written asymptotically as a sum of $\varepsilon_{\mathcal{U}_{m,k}(i)}$, by imposing a supplementary condition:

(A6) for any $\epsilon > 0$ there exists $M > 0$ such that $\mathbf{P} \left[\max_{1 \leq i \leq m} \|\dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)\|_2 \geq M \right] \leq \epsilon$.

The proof of Proposition 4.2 is given in Section 6, where the nonlinearity intervenes decisively to prove that the sum of 1 to T_m for the right-hand side of an expression like (12) converges uniformly in probability to zero.

Proposition 4.2 *Under the assumptions (A1)-(A4), (A6) we have for any $\epsilon > 0$, in probability,*

$$\sup_{1 \leq k < \infty} \mathbf{P}_{m,k}^* \left[\max_{1 \leq l \leq T_m} \frac{|I_2 - (-l/m \sum_{j=1}^m \varepsilon_{\mathcal{U}_{m,k}(j)})|}{g(m, l, \gamma)} \geq \epsilon \right] \xrightarrow{m \rightarrow \infty} 0.$$

Taking into account the proof of Theorem 3.1 concerning the asymptotic distribution of the weighted cumulative residuals sum $\Gamma(m, k, \gamma)$ calculated without bootstrapping, we show by the following results that the statistic $\tilde{\Gamma}(m, k, l, \gamma)$ bootstrapped has the same asymptotic behavior under H_0 as $\Gamma(m, k, \gamma)$. Under hypothesis H_1 , the term \mathcal{R}_m is asymptotically uniformly bounded and then, taking into account the relation (13), $\tilde{\Gamma}(m, k, l, \gamma)$ is uniformly bounded a.s. also (see in Appendix, sub-Section 7.2, the Lemmas 7.3 and 7.4).

Proposition 4.3 *Suppose that the assumptions (A1)-(A4), (A6) hold.*

a) *Under the null hypothesis H_0 , we have, for any $x \in \mathbb{R}$,*

$$\sup_{1 \leq k \leq T_m} \left| \mathbf{P}_{m,k}^* \left[\frac{1}{\hat{\sigma}_{m,k}} \sup_{1 \leq l \leq T_m} \tilde{\Gamma}(m, k, l, \gamma)(\varepsilon_{m,k}^*(1), \dots, \varepsilon_{m,k}^*(m+l)) \leq x \right] - \mathbf{P} \left[\sup_{1 \leq l \leq T_m} \frac{|W_1(\frac{l}{m}) - \frac{l}{m} \mathcal{D}W_2(1)|}{(1 + \frac{l}{m}) \left(\frac{l}{m+l}\right)^\gamma} \leq x \right] \right| \xrightarrow{m \rightarrow \infty} 0.$$

where $\{W_1(t); 0 \leq t < \infty\}$ is a Wiener process, $W_2(1)$ is a standard normally distributed, independent of $\{W_1(t)\}$.

b) *If furthermore the assumption (A5) holds, under the alternative hypothesis H_1 , for any $\epsilon > 0$, there exists a constant $M > 0$ such that, we have a.s.*

$$\sup_{1 \leq k \leq T_m} \mathbf{P}_{m,k}^* \left[\frac{1}{\hat{\sigma}_{m,k}} \sup_{1 \leq l \leq T_m} |\tilde{\Gamma}(m, k, l, \gamma)(\varepsilon_{m,k}^*(1), \dots, \varepsilon_{m,k}^*(m+l))| \geq M \right] \leq \epsilon + o_{\mathbf{P}}(1).$$

As for the Theorem 3.1, under H_0 , we can prove that the asymptotic distribution of $\sup_{1 \leq l \leq T_m} \frac{|W_1(l/m) - l/m \mathcal{D}W_2(1)|}{(1+l/m)(l/(m+l))^\gamma}$ is $\sup_{0 \leq t \leq \frac{1}{\mathcal{D}^2}} \frac{(1+t-\mathcal{D}^2 t)|W(t)|}{t^\gamma}$ in the case $T_m = \infty$ or ($T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = \infty$). In the case $T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = T < \infty$, the asymptotic distribution is $\sup_{0 \leq t \leq \frac{T}{1+\mathcal{D}^2 T}} \frac{(1+t-\mathcal{D}^2 t)|W(t)|}{t^\gamma}$. Combining Theorem 3.1 with Proposition 4.3(a) under the null hypothesis, on the one hand, and Theorem 3.2 with Proposition 4.3(b) under the alternative hypothesis, on the other hand, together with the distribution function definition $\tilde{F}_{m,k}$, allow to define a critical value depending of each sequential observation $k = 1, \dots, T_m$. Thus, we can define a new test statistic and study its asymptotic behavior under H_0 and H_1 .

Theorem 4.1 *Suppose that the assumptions (A1)-(A4), (A6) hold and that $\alpha \in (0, 1)$, $\gamma \in [0, 1/2)$.*

a) *Under the null hypothesis H_0 , as $m \rightarrow \infty$, we have*

$$P \left[\frac{1}{\hat{\sigma}_m} \sup_{1 \leq k \leq T_m} \frac{|\Gamma(m, k, \gamma)|}{c_{m,k;\alpha}(\gamma)} > 1 \right] \rightarrow \alpha.$$

b) *If furthermore the assumption (A5) holds, under the alternative hypothesis H_1 , as $m \rightarrow \infty$, we have*

$$P \left[\frac{1}{\hat{\sigma}_m} \sup_{1 \leq k \leq T_m} \frac{|\Gamma(m, k, \gamma)|}{c_{m,k;\alpha}(\gamma)} > 1 \right] \rightarrow 1,$$

with Γ given by the relation (3) and $c_{m,k;\alpha}(\gamma)$ by (9) is the critical value of the distribution function $\tilde{F}_{m,k}$.

Thus, we are going to use as test statistic of H_0 , against H_1

$$Z_{\gamma;\alpha}^{(b)}(m) \equiv \frac{1}{\hat{\sigma}_m} \sup_{1 \leq k \leq T_m} \frac{|\Gamma(m, k, \gamma)|}{c_{m,k;\alpha}(\gamma)}, \quad (14)$$

which will have the asymptotic critical region $\{Z_{\gamma;\alpha}^{(b)}(m) > 1\}$. Then the statistic $Z_{\gamma;\alpha}^{(b)}(m)$ has asymptotic size α and asymptotic power one for all $\gamma \in [0, 1/2)$. As in Section 4, we consider the change-point estimator of k_m^0 is

$$\hat{\tau}_m^{(b)} \equiv \begin{cases} \inf \left\{ 1 \leq k \leq T_m, \frac{|\Gamma(m, k, \gamma)|}{\hat{\sigma}_m c_{m,k;\alpha}(\gamma)} > 1 \right\}, \\ \infty, \text{ if } \frac{|\Gamma(m, k, \gamma)|}{\hat{\sigma}_m c_{m,k;\alpha}(\gamma)} \leq 1, \text{ for every } 1 \leq k \leq T_m. \end{cases} \quad (15)$$

Then, hypothesis H_0 is rejected in $\hat{\tau}_m^{(b)}$. Let us notice that, in comparison with the previous test, the value calculation $c_{m,k;\alpha}(\gamma)$ is little more laborious, in view of the fact that, the conditional distribution functions $F_{m,k}^*$ must be first calculated.

5. Simulations

In this section we report a simulation study designed to evaluate and compare the performance of the proposed test methods. For the two methods we consider two examples: growth model and compartmental

model for varied parameters, sample size or position of k_m^0 after m . For each test statistic, the algorithm steps are given to calculate the corresponding critical values. Afterward, details are given how to calculate empirical test size, empirical test power and to estimate the change-point location. All simulations were performed using the R language. The program codes can be requested from the author.

5.1. Test by weighted CUSUM, without bootstrapping

Firstly, following simulation steps are realized in order to calculate the critical values $c_\alpha(\gamma)$ in accordance with the Corollary 3.1:

1. Calculate $\mathcal{D} \equiv [\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}]^{1/2}$.
2. Simulate M replications of the random variable $V_\gamma = \sup_{0 \leq t \leq 1/\mathcal{D}^2} \frac{(1+t-\mathcal{D}^2 t)|W(t)|}{t^\gamma}$, with $\{W(t), 0 \leq t \leq 1/\mathcal{D}^2\}$ a Wiener process, or $V_\gamma = \sup_{0 \leq t \leq \frac{T}{1+\mathcal{D}^2 T}} \frac{(1+t-\mathcal{D}^2 t)|W(t)|}{t^\gamma}$, with $\{W(t), 0 \leq t \leq T/1 + \mathcal{D}^2 T\}$ a Wiener process, respectively, taking into account the two possible cases (i) or (ii) concerning T_m of Theorem 3.1.
3. On the basis of M replications of V_γ we calculate the critical values $c_\alpha(\gamma)$ such that $P[V_\gamma > c_\alpha(\gamma)] = \alpha$.

A Brownian motion is generated using the *BM* function in R package(*sde*). Once the critical values $c_\alpha(\gamma)$ are available, the change absence against the change of the model is tested using the statistic $Z_\gamma(m)$, given by relation (7). In order to calculate the empirical test size, an without change-point model is considered and we count, the number of times, on the Monte-Carlo replications, when we obtain $Z_\gamma(m) > c_\alpha(\gamma)$. For the calculation of the empirical test power, the hypothesis H_1 is considered true, that there exists a change-point. We fix $T_m = 500$, $k_m^0 = 25$ (or $k_m^0 = 2$) and we vary the sample size $m = 25, 100, 300$, $\gamma = 0, 0.25, 0.45, 0.49$, $\alpha = 0.025, 0.05, 0.10$. For every combination, 1000 Monte-Carlo replications are realized. On the 1000 replications, we computed the frequency among which the test statistic $Z_\gamma(m)$ exceeds the critical value $c_\alpha(\gamma)$. In order to estimate the change-point location, we find the first point k in the interval $1, \dots, T_m$ such that $(\hat{\sigma}_m)^{-1} |\sum_{i=m+1}^{m+k} \hat{\varepsilon}_i| / g(m, k, \gamma)$ exceeds critical value $c_\alpha(\gamma)$.

For both models, in order to study the importance that $\ddot{\mathbf{f}}(\mathbf{x}, \boldsymbol{\beta}_m^0)$ is bounded or not, two regression parameters $\boldsymbol{\beta}_m^0$ after the change-point are considered: one for which $\ddot{\mathbf{f}}(\mathbf{x}, \boldsymbol{\beta}_m^0)$ is bounded and another for which $\ddot{\mathbf{f}}(\mathbf{x}, \boldsymbol{\beta}_m^0)$ is not bounded. Even though the theoretical results are valid, we will study the precision of the change-point location estimator.

5.1.1. Growth model

Let us consider first the growth function $f(x; \boldsymbol{\beta}) = b_1 - \exp(-b_2 x)$ which models many phenomena, with the parameters $\boldsymbol{\beta} = (b_1, b_2) \in \Theta$, $\Theta \subseteq \mathbb{R} \times \mathbb{R}_+$ compact and $x \in \mathbb{R}$. In this case the dimension of $\boldsymbol{\beta}$ is 2 ($q = 2$) and it there is a single regressor ($p = 1$). We generate the response variable $X \sim \mathcal{N}(0, \sigma_X^2)$ and the errors $\varepsilon \sim \mathcal{N}(0, 0.5)$. The true values of regression parameters before the change-point are $\boldsymbol{\beta}^0 = (0.5, 1)$ and after $\boldsymbol{\beta}_m^0 = (1, 2)$. By elementary calculations we obtain $E[X \exp(-b_2 X)] = -b_2 \sigma_X^2 \exp(b_2^2 \sigma_X^2 / 2)$, $E[X^2 \exp(-2b_2 X)] = \sigma_X^2 [1 + 4b_2^2] \exp(2b_2^2 \sigma_X^2)$, then

$$\mathbf{A} = E[\dot{\mathbf{f}}(X, \boldsymbol{\beta})] = \begin{bmatrix} 1 \\ -b_2 \sigma_X^2 \exp(b_2^2 \sigma_X^2 / 2) \end{bmatrix}$$

Table 1: The $(1 - \alpha)$ quantiles (critical values) $c_\alpha(\gamma)$ of the random variable V_γ (specified in subsection 5.1, Step 2) calculated on 50000 Monte-Carlo replications. Growth model.

$\gamma \downarrow; \alpha \rightarrow$	0.01	0.025	0.05	0.10	0.25
0	2.7959	2.5033	2.2411	1.9595	1.5322
0.15	2.8581	2.5690	2.3058	2.0313	1.6146
0.25	2.9243	2.6368	2.3841	2.1082	1.7014
0.35	3.0220	2.7536	2.5044	2.2414	1.8462
0.45	3.2578	3.0051	2.7878	2.5391	2.1639
0.49	3.5214	3.2668	3.0473	2.8040	2.4133

Table 2: Empirical sizes of test based on the statistic (7) for a growth model. Calculated for 1000 Monte-Carlo replications and $T_m = 500$.

$\gamma \downarrow$	$\alpha = 0.025$			$\alpha = 0.05$			$\alpha = 0.10$		
	m=25	m=100	m=300	m=25	m=100	m=300	m=25	m=100	m=300
0	0.0051	0.0026	0.0003	0.0075	0.0038	0.0006	0.0132	0.0081	0.0020
0.25	0.0058	0.0025	0.0023	0.0084	0.0043	0.0043	0.0154	0.0075	0.0079
0.45	0.0066	0.0033	0.0023	0.0089	0.0050	0.0043	0.0130	0.089	0.0079
0.49	0.0046	0.0021	0.0014	0.0065	0.0032	0.0026	0.0093	0.0064	0.0070

$$\mathbf{B} = E[\dot{\mathbf{f}}(X; \boldsymbol{\beta}) \dot{\mathbf{f}}^t(X; \boldsymbol{\beta})] = \begin{bmatrix} 1 & -b_2 \sigma_X^2 \exp(b_2^2 \sigma_X^2 / 2) \\ -b_2 \sigma_X^2 \exp(b_2^2 \sigma_X^2 / 2) & \sigma_X^2 [1 + 4b_2^2] \exp(2b_2^2 \sigma_X^2) \end{bmatrix}.$$

Obviously $\mathcal{D} = 1$ for any value of σ_X^2 and of the parameters b_1, b_2 . This means that we obtain the same quantiles that in the paper of the Horváth et al.(2004). The empirical quantiles (critical values) $c_\alpha(\gamma)$ of the random variable V_γ are given in the Table 1. Based on these empirical quantiles, we are going to study the test size and its power for various values of m, γ and α . We realize 1000 Monte-Carlo replication of the model and we take $T_m = 500$. The empirical test sizes are presented in Table 2. We observe that the obtained values are smaller widely to the fixed α theoretical size. On the 1000 replications we found that empirical test power is 1, in any case. For the same parameters, we estimate now as follows the change-point location. For $\gamma = 0.49, 0.25, \gamma = 0$ and $m = 25$ or 100, after 10000 Monte-Carlo model replications in Table 3 are given the minimum, median, mean, third quartile and maximum of the change-point location estimations. For $m = 300$, the results are similar to those obtained for $m = 100$, thus we don't present them. We observe that the obtained change-point estimates are biased, and that considering either the median or the mean, there is a delay time in change-point detection. In the Table 4 we have the summarized results when the change-point is immediately later after m , for $k_m^0 = 2$. From these two Tables 3 and 4 we deduce that, with respect to γ , when the change is in $k_m^0 = 25$, there is no difference concerning the location change-point precision. If the change is immediately ($k_m^0 = 2$), the precision decreases when γ decreases.

In all tables, we indicated between "()" the obtained results when $\boldsymbol{\beta}_m^0 = (1, -0.5)$, case in which the function $\ddot{\mathbf{f}}(\mathbf{x}; \boldsymbol{\beta}_m^0)$ is not bounded for all \mathbf{x} . The results are worse, even though the break in k_m^0 is largest.

Table 3: Estimation of the change-point location based on the statistic (7), for 10000 Monte-Carlo replications, $T_m = 500$, $k_m^0 = 25$, $\beta^0 = (0.5, 1)$, $\beta_m^0 = (1, 2)$ and between () for $\beta_m^0 = (1, -0.5)$. Growth model.

		$m=25$			$m=100$		
γ	$\text{summary}(\hat{\tau}_m) \downarrow; \alpha \rightarrow$	0.025	0.05	0.10	0.025	0.05	0.10
0.49	<i>min</i>	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
	<i>median(Q2)</i>	32 (41)	32 (39)	31 (37)	31 (37)	31 (35)	31 (34)
	<i>mean</i>	35 (46)	34 (43)	33 (40)	34 (39)	33 (38)	32 (36)
	<i>Q3</i>	39 (54)	38 (51)	37 (48)	37 (45)	37 (43)	36 (41)
	<i>max</i>	148 (279)	148 (248)	148 (232)	112 (144)	111 (143)	109 (132)
0.25	<i>min</i>	1 (1)	1 (1)	1 (1)	9 (1)	9 (1)	9 (1)
	<i>median(Q2)</i>	32 (39)	31 (37)	31 (35)	32 (38)	32 (37)	31 (35)
	<i>mean</i>	34 (43)	33 (40)	32 (37)	34 (41)	34 (39)	33 (38)
	<i>Q3</i>	39 (49)	38 (46)	36 (43)	38 (46)	37 (44)	37 (42)
	<i>max</i>	141 (245)	141 (219)	119 (200)	110 (137)	110 (125)	95 (125)
0	<i>min</i>	1 (1)	1 (1)	1 (1)	9 (25)	9 (21)	9 (20)
	<i>median(Q2)</i>	33 (41)	32 (39)	31 (36)	34 (43)	33 (41)	33 (39)
	<i>mean</i>	35 (45)	34 (42)	33 (39)	36 (45)	35 (43)	35 (41)
	<i>Q3</i>	39 (52)	38 (49)	37 (45)	41 (52)	40 (49)	39 (46)
	<i>max</i>	121 (300)	115 (284)	115 (223)	115 (136)	115 (132)	115 (131)

Table 4: Estimation of the change-point location based on the statistic (7), for 10000 Monte-Carlo replications, $T_m = 500$, $k_m^0 = 2$, $\beta^0 = (0.5, 1)$, $\beta_m^0 = (1, 2)$ and between () for $\beta_m^0 = (1, -0.5)$. Growth model.

		$m=25$			$m=100$		
γ	$\text{summary}(\hat{\tau}_m) \downarrow; \alpha \rightarrow$	0.025	0.05	0.10	0.025	0.05	0.10
0.49	<i>min</i>	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
	<i>median(Q2)</i>	6 (6)	6 (6)	5 (5)	6 (6)	6 (6)	5 (5)
	<i>mean</i>	8 (10)	8 (9)	7 (8)	8 (9)	7 (8)	7 (7)
	<i>Q3</i>	10 (12)	10 (11)	9 (10)	10 (11)	9 (10)	9 (9)
	<i>max</i>	109 (224)	109 (218)	79 (185)	91 (81)	91 (71)	91 (67)
0.25	<i>min</i>	1 (1)	1 (1)	1 (1)	1 (2)	1 (1)	1 (1)
	<i>median(Q2)</i>	7 (8)	6 (7)	6 (6)	7 (10)	7 (9)	7 (8)
	<i>mean</i>	9 (11)	8 (10)	7 (9)	10 (12)	9 (11)	9 (10)
	<i>Q3</i>	11 (14)	10 (12)	10 (11)	13 (16)	12 (15)	11 (13)
	<i>max</i>	74 (156)	72 (133)	72 (133)	89 (94)	87 (94)	85 (94)
0	<i>min</i>	1 (1)	1 (1)	1 (1)	6 (3)	6 (3)	5 (3)
	<i>median(Q2)</i>	8 (11)	7 (10)	7 (9)	10 (17)	9 (15)	9 (14)
	<i>mean</i>	10 (14)	9 (13)	9 (11)	12 (19)	12 (17)	11 (15)
	<i>Q3</i>	13 (18)	12 (16)	11 (14)	16 (25)	15 (23)	15 (20)
	<i>max</i>	93 (210)	88 (179)	88 (175)	91 (131)	76 (124)	73 (111)

5.1.2. Compartmental model

Another very interesting nonlinear model, with numerous applications, is the compartmental model. Examples and references of important applications for these models are given in Seber and Wild(2003) (see also the references therein): it describes the movement of lead in the human body, the kinetics of drug movement when the drug is injected at an intramuscular site, etc... Consider two-compartment function $h_\beta(x) = b_1 \exp(-b_1 x) + b_2 \exp(-b_2 x)$, $\beta = (b_1, b_2) \in \Theta \subseteq \mathbb{R}_+^2$. In this case $q = 2$ and $p = 1$.

As for the growth example, we consider a gaussian response variable $X \sim \mathcal{N}(0, \sigma_X^2)$. For this model we have $E[\exp(-bX)] = \exp(b^2 \sigma_X^2 / 2)$, $E[X \exp(-bX)] = -b \sigma_X^2 \exp(b^2 \sigma_X^2 / 2)$, $E[X^2 \exp(-2bX)] = \sigma_X^2 [1 + 4b^2] \exp(2b^2 \sigma_X^2)$. Then

$$\mathbf{A} = E[\dot{\mathbf{f}}(X; \beta)] = \begin{bmatrix} (1 + b_1 \sigma_X^2) \exp(b_1^2 \sigma_X^2 / 2) \\ -(1 + b_2 \sigma_X^2) \exp(b_2^2 \sigma_X^2 / 2) \end{bmatrix}.$$

And with the notations $B_{11} = 1 + b_1^2 \sigma_X^2 (5 + 4b_1^2) \exp(2b_1^2 \sigma_X^2)$, $B_{12} = 1 + \sigma_X^2 [(b_1 + b_2)^2 + b_1 b_2 (1 + (b_1 + b_2)^2)] \exp((b_1 + b_2)^2 \sigma_X^2 / 2)$, $B_{22} = 1 + b_2^2 \sigma_X^2 (5 + 4b_2^2) \exp(2b_2^2 \sigma_X^2)$, we have the matrix

$$\mathbf{B} = E[\dot{\mathbf{f}}(X; \beta) \dot{\mathbf{f}}^t(X; \beta)] = \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix}.$$

Contrary to the previous case, the value of \mathcal{D} depends on the variance σ_X^2 of the random variable X and on the parameters of the growth function. Hence, for each value of β^0 and of variance of X we need to calculate the quantiles. For the simulations, let us consider $\sigma_X^2 = 1$ and $\beta^0 = (1.2, 1)$. In this case $\mathcal{D} = 0.5741$.

The empirical quantiles $c_\alpha(\gamma)$ of the random variable V_γ , specified at the beginning of this subsection, are given in the Table 5.

The simulations are carried out for historical data of size $m = 25, 100$ or 300 and $T_m = 500$ observation after m . The empirical type I error probabilities are presented in the Table 6 calculated by 1000 Monte-Carlo replications. As for the growth example, the empirical power test is 1 for each value of γ, α , when $k_m^0 = 25$ and $\beta_m^0 = (1, 2)$.

In Tables 7 and 8, the summarized results on the change-point estimations obtained on 10000 Monte-Carlo replications, varying m, γ and theoretic test size α . Between "()" we give the results for $\beta_m^0 = (-0.5, 2)$, when the function $\dot{\mathbf{f}}(\mathbf{x}; \beta_m^0)$ is not bounded for all value of \mathbf{x} .

We can make the following observations. As for the growth example, the results are less good in the case $\dot{\mathbf{f}}(\mathbf{x}; \beta)$ not bounded: the method detects later the change and especially we have greater maximal values for the change-point estimation $\hat{\tau}_m$. In the two case, $k_m^0 = 25$ and $k_m^0 = 2$, the precision of $\hat{\tau}_m$ decreases when γ decreases. The change-point estimation is more precise than for the growth model.

5.2. Test using the bootstrapping

In this case, the calculation of the critical values $c_{m,k;\alpha}(\gamma)$ defined by (9) is more laborious. We go to see if the simulation results are better than by weighted CUSUM without bootstrapping, case in which it

Table 5: The $(1 - \alpha)$ quantiles (critical values) $c_\alpha(\gamma)$ of the random variable V_γ (specified in subsection 5.1, Step 2) calculated on 50000 Monte-Carlo replications. Compartmental model, $\beta^0 = (1.2, 1)$, $\sigma_X^2 = 1$.

$\gamma \downarrow ; \alpha \rightarrow$	0.01	0.025	0.05	0.10	0.25
0	6.2165	5.5233	4.9211	4.2812	3.2689
0.15	5.7627	5.1279	4.5862	4.0014	3.0854
0.25	5.4929	4.9022	4.3838	3.8395	2.9833
0.35	5.2355	4.6960	4.2092	3.7024	2.9142
0.45	5.0383	4.5223	4.0786	3.5998	2.9191
0.49	4.9682	4.4702	4.0555	3.6032	2.9945

Table 6: Empirical sizes of test based on the statistic (7) for a compartmental model. Calculated for 1000 Monte-Carlo replications and $T_m = 500$.

γ	$\alpha = 0.025$			$\alpha = 0.05$			$\alpha = 0.10$		
	m=25	m=100	m=300	m=25	m=100	m=300	m=25	m=100	m=300
0	0.0003	0	0	0.0005	0	0	0.0007	0	0
0.25	0.0006	0	0	0.0006	0	0	0.0009	0	0
0.45	0.0010	0	0	0.0013	0	0	0.0014	0.0002	0
0.49	0.0009	0	0	0.0011	0.0001	0.0001	0.0012	0.0004	0.0002

Table 7: Estimation of the change-point location based on the statistic (7), for 10000 Monte-Carlo replications, $T_m = 500$, $k_m^0 = 25$, $\beta^0 = (1.2, 1)$, $\beta_m^0 = (1, 2)$ and between () for $\beta_m^0 = (-0.5, 2)$. Compartmental model.

γ	$summary(\hat{\tau}_m) \downarrow ; \alpha \rightarrow$	$m = 25$			$m = 100$		
		0.025	0.05	0.10	0.025	0.05	0.10
0.49	min	1 (1)	1 (1)	1 (1)	26 (1)	5 (1)	1 (1)
	median(Q2)	30 (31)	30 (31)	29 (31)	29 (31)	29 (30)	29 (30)
	mean	30 (34)	30 (33)	30 (33)	30 (32)	30 (32)	29 (32)
	Q3	33 (37)	33 (37)	32 (36)	32 (36)	32 (35)	31 (34)
	max	63 (105)	63 (102)	60 (102)	55 (105)	53 (105)	53 (102)
0.25	min	1 (1)	1 (1)	1 (1)	26 (26)	26 (26)	26 (26)
	median(Q2)	31 (32)	30 (32)	30 (31)	30 (32)	30 (32)	30 (31)
	mean	31 (35)	31 (34)	30 (34)	31 (34)	31 (34)	30 (33)
	Q3	34 (39)	34 (38)	33 (37)	34 (38)	34 (38)	33 (37)
	max	69 (124)	69 (124)	59 (115)	61 (99)	60 (99)	52 (97)
0	min	3 (1)	3 (1)	3 (1)	26 (26)	26 (26)	26 (26)
	median(Q2)	32 (34)	31 (33)	31 (33)	32 (35)	32 (34)	31 (33)
	mean	33 (37)	32 (36)	32 (35)	33 (37)	33 (36)	32 (35)
	Q3	36 (41)	35 (40)	34 (39)	37 (42)	36 (41)	35 (40)
	max	73 (143)	67 (138)	63 (114)	69 (120)	69 (120)	66 (102)

Table 8: Estimation of the change-point location based on the statistic (7), for 10000 Monte-Carlo replications, $T_m = 500$, $k_m^0 = 2$, $\beta^0 = (1.2, 1)$, $\beta_m^0 = (1, 2)$ and between () for $\beta_m^0 = (-0.5, 2)$. Compartmental model.

γ	summary($\hat{\tau}_m$) \downarrow ; $\alpha \rightarrow$	$m = 25$			$m = 100$		
		0.025	0.05	0.10	0.025	0.05	0.10
0.49	<i>min</i>	1 (1)	1 (1)	1 (1)	3 (3)	3 (2)	3 (1)
	<i>median(Q2)</i>	5 (6)	4 (5)	4 (5)	5 (5)	4 (5)	4 (5)
	<i>mean</i>	5 (7)	5 (7)	5 (6)	5 (7)	5 (6)	5 (6)
	<i>Q3</i>	7 (9)	6 (8)	6 (8)	7 (8)	6 (8)	6 (7)
	<i>max</i>	29 (68)	29 (61)	29 (60)	27 (80)	27 (80)	27 (49)
0.25	<i>min</i>	1 (1)	1 (1)	1 (1)	3 (3)	3 (3)	3 (3)
	<i>median(Q2)</i>	5 (7)	5 (6)	5 (6)	6 (7)	6 (7)	5 (7)
	<i>mean</i>	6 (9)	6 (8)	6 (8)	7 (9)	6 (9)	6 (9)
	<i>Q3</i>	8 (11)	8 (11)	7 (10)	9 (12)	9 (12)	8 (11)
	<i>max</i>	37 (86)	37 (86)	35 (86)	36 (98)	35 (97)	34 (90)
0	<i>min</i>	1 (1)	1 (1)	1 (1)	3 (3)	3 (3)	3 (3)
	<i>median(Q2)</i>	7 (8)	6 (8)	6 (8)	9 (11)	9 (10)	8 (9)
	<i>mean</i>	7 (11)	7 (10)	7 (10)	9 (13)	9 (12)	8 (12)
	<i>Q3</i>	10 (14)	9 (14)	9 (13)	13 (16)	12 (15)	11 (15)
	<i>max</i>	35 (99)	34 (91)	34 (91)	49 (110)	40 (91)	40 (91)

deserves to make calculation effort.

We now describe in detail the algorithm steps for calculate the critical values $c_{m,k;\alpha}(\gamma)$.

Step 1. We fix $\alpha, \gamma, N, L, m, T_m$ (see the notations given in Section 4 for N and L).

Step 2.

- We calculate $J = T_m/L$;
- For $j = 0, 1, \dots, (J-1)L$, the following random variable are generated

$$\tilde{V}_j \equiv \frac{1}{\hat{\sigma}_{m,j}^{(*)}} \sup_{1 \leq l \leq T_m} |\tilde{\Gamma}(m, j, l, \gamma)(\varepsilon_{m,j}^*(1), \dots, \varepsilon_{m,j}^*(m+l))|$$

Step 3. For $\tilde{j} = 0, 1, \dots, (J-1)L$, we generate the random variables $\tilde{W}_{\tilde{j}}$ which are mixtures of the random variables \tilde{V}_j generated to step 2.

For each $\tilde{j} = 1, \dots, (J-1)L$, we generate a multinomial distribution with parameters 1(number of trials) and the probability vector $p_{\tilde{j}} = (1/\tilde{j}, \dots, 1/\tilde{j})$. On the basis of this, thus, $\tilde{W}_{\tilde{j}} = \tilde{V}_j$ for $j = 0, 1, \dots, \tilde{j} - 1$ with the probability $1/\tilde{j}$.

Step 4. We repeat the steps 2 and 3 making M Monte-Carlo replications. At the end, we shall have M realizations for every random variable $\tilde{W}_{\tilde{j}}$, $\tilde{j} = 0, 1, \dots, (J-1)L$.

Step 5. We calculate for every $k = jL, jL+1, \dots, (j+1)L-1$ for $j = 0, 1, \dots, J$ the random variables $\tilde{U}_k = \tilde{W}_j$.

Step 6. On the basis of M replications, for each $k = 1, \dots, T_m$, we calculate the critical values $c_{m,k;\alpha}(\gamma)$ such

Table 9: Empirical sizes of test based on the statistic \tilde{U}_k given in subsection 5.2, Step 5, for a compartmental model, for bootstrap critical values. Calculated for 1000 Monte-Carlo replications and $T_m = 500$.

γ	$\alpha = 0.025$		$\alpha = 0.05$		$\alpha = 0.10$	
	m=25	m=100	m=25	m=100	m=25	m=100
0.25	0.0001	0	0.0003	0	0.0007	0
0.49	0.0004	0	0.0006	0	0.0008	0

that $P[\tilde{U}_k > c_{m,k;\alpha}(\gamma)] = \alpha$.

The change absence against the change of the model is tested using the statistic $Z_{\gamma,\alpha}^{(b)}(m)$ given by (14). In order to calculate the empirical test size, an without change-point model is considered and we count, the number of times, on the Monte Carlo replications, when we obtain $Z_{\gamma,\alpha}^{(b)}(m) > 1$. Recall that the change-point estimation $\hat{\tau}_m^{(b)}$ is calculated using relation (15).

Let us consider $m = 25$ and $m = 100$. For $m = 300$, the results are similar to those obtained for $m = 100$, thus we don't present them. In the case $m = 100$ we consider $L = m/50$ and in the case $m = 25$ we take $L = m/10$. For γ we take only two values: 0.25 et 0.49. If $k_m^0 = 25$, the empirical power test is 1 in all cases: for the two model type (growth or compartmental) and for the every parameters γ and k_m^0 .

The same parameter settings are used as in the previous simulation study, in the subsection 5.1.

5.2.1. Compartmental model

The empirical test size based on the statistic \tilde{U}_k (of Step 5), calculated for 1000 Monte-Carlo replications and $T_m = 500$, are given in the Table 9. By comparing the Tables 6 and 9, we deduce that the empirical test sizes are smaller by the bootstrapping method.

The results concerning $\hat{\tau}_m^{(b)}$, the estimation of k_m^0 , presented in the Tables 10 and 11, are almost the same for $\gamma = 0.49$ and $\gamma = 0.25$. Apart from $\gamma = 0.25$ and $k_m^0 = 25$, the results for $\hat{\tau}_m^{(b)}$ are not better than those obtained by the method without bootstrapping.

5.2.2. Growth model

Tables 2 and 12 indicate that the empirical test size obtained using the bootstrapped critical values are sharply lower than empirical test size without bootstrapping. Concerning the change-point estimation (Table 13), for $\gamma = 0.49$, $m = 25$ and $k_m^0 = 25$, the results for $\hat{\tau}_m^{(b)}$ are better than by the weighted CUSUM method without bootstrapping. On the other hand, for $\gamma = 0.49$, $m = 100$, the results are less good using the bootstrapped critical values.

5.3. Conclusion on the simulations

Two test statistics and their critical regions are, using weighted CUSUM method without and with bootstrapping for two nonlinear models. In both cases, the empirical sizes are widely smaller than the fixed theoretical size α . But the empirical sizes of test are without thinking smaller when the critical values are

Table 10: Estimation of the change-point location based on the statistic (14), for 10000 Monte-Carlo replications, $T_m = 500$, $k_m^0 = 25$, $\beta^0 = (1.2, 1)$, $\beta_m^0 = (1, 2)$ and between () for $\beta_m^0 = (-0.5, 2)$. Compartmental model.

γ	$summary(\hat{\tau}_m^{(b)}) \downarrow; \alpha \rightarrow$	$m=25$			$m=100$		
		0.025	0.05	0.10	0.025	0.05	0.10
0.49	<i>min</i>	2 (2)	1 (1)	1 (1)	26 (26)	26 (26)	14 (26)
	<i>median(Q2)</i>	32 (30)	30 (28)	26 (27)	32 (35)	31 (33)	29 (32)
	<i>mean</i>	32 (33)	30 (38)	27 (29)	32 (37)	31 (36)	30 (34)
	<i>Q3</i>	36 (36)	33 (35)	30 (33)	35 (42)	34 (40)	32 (38)
	<i>max</i>	90 (118)	78 (118)	65 (92)	72 (122)	63 (122)	54 (122)
0.25	<i>min</i>	5 (1)	5 (1)	1 (1)	26 (26)	26 (26)	26 (26)
	<i>median(Q2)</i>	28 (30)	26 (28)	24 (26)	32 (35)	31 (34)	30 (32)
	<i>mean</i>	28 (32)	27 (30)	25 (28)	32 (38)	32 (36)	31 (34)
	<i>Q3</i>	32 (38)	28 (34)	28 (31)	34 (41)	34 (40)	34 (38)
	<i>max</i>	66 (106)	59 (103)	57 (103)	71 (134)	66 (118)	57 (98)

Table 11: Estimation of the change-point location based on the statistic (14), for 10000 Monte-Carlo replications, $T_m = 500$, $k_m^0 = 2$, $\beta^0 = (1.2, 1)$, $\beta_m^0 = (1, 2)$ and between () for $\beta_m^0 = (-0.5, 2)$. Compartmental model.

$\gamma = 0.49$	$summary(\hat{\tau}_m^{(b)}) \downarrow; \alpha \rightarrow$	$m=25$			$m=100$		
		0.025	0.05	0.10	0.025	0.05	0.10
0.49	<i>min</i>	3 (1)	2 (1)	2 (1)	3 (3)	3 (3)	3 (3)
	<i>median(Q2)</i>	6 (8)	6 (6)	5 (5)	6 (7)	5 (7)	5 (6)
	<i>mean</i>	8 (11)	7 (9)	6 (7)	7 (10)	6 (10)	5 (8)
	<i>Q3</i>	10 (13)	9 (11)	8 (9)	9 (14)	7 (12)	7 (10)
	<i>max</i>	65 (76)	64 (76)	33 (63)	43 (86)	42 (75)	27 (75)
0.25	<i>min</i>	3 (2)	1 (1)	1 (1)	3 (3)	3 (3)	3 (3)
	<i>median(Q2)</i>	6 (7)	5 (6)	4 (5)	7 (10)	6 (10)	6 (7)
	<i>mean</i>	7 (10)	6 (9)	5 (7)	8 (13)	7 (12)	7 (9)
	<i>Q3</i>	9 (13)	8 (12)	7 (9)	11 (18)	10 (16)	9 (12)
	<i>max</i>	38 (81)	32 (81)	28 (62)	37 (119)	36 (74)	32 (74)

Table 12: Empirical sizes of test based on the statistic \tilde{U}_k given in subsection 5.2, Step 5, for a growth model, for bootstrap critical values. Calculated for 1000 Monte-Carlo replications and $T_m = 500$.

γ	$\alpha = 0.025$		$\alpha = 0.05$		$\alpha = 0.10$	
	m=25	m=100	m=25	m=100	m=25	m=100
0.25	0.0005	0	0.0012	0	0.0027	0.0004
0.49	0.0002	0	0.0003	0	0.0007	0

Table 13: Estimation of the change-point location based on the statistic (14), for 10000 Monte-Carlo replications, $T_m = 500$, $\gamma = 0.49$, $\beta^0 = (0.5, 1)$, $\beta_m^0 = (1, 2)$ and between () for $\beta_m^0 = (1, -0.5)$. Growth model.

k_m^0	$summary(\hat{z}_m^{(b)}) \downarrow ; \alpha \rightarrow$	$m=25$			$m=100$		
		0.025	0.05	0.10	0.025	0.05	0.10
25	<i>min</i>	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
	<i>median(Q2)</i>	27 (32)	26 (30)	25 (28)	38 (53)	36 (45)	34 (42)
	<i>mean</i>	28 (35)	27 (33)	26 (30)	40 (57)	39 (51)	36 (44)
	<i>Q3</i>	32 (39)	32 (39)	30 (36)	48 (61)	44 (60)	42 (50)
	<i>max</i>	107 (242)	107 (182)	107 (175)	146 (220)	145 (184)	140 (156)
2	<i>min</i>	1 (1)	1 (1)	(1) 1	1 (2)	1 (1)	1 (1)
	<i>median(Q2)</i>	5 (6)	5 (5)	5 (5)	8 (11)	8 (10)	7 (9)
	<i>mean</i>	7 (8)	7 (8)	6 (7)	11 (19)	10 (15)	9 (12)
	<i>Q3</i>	9 (10)	8 (9)	8 (8)	14 (26)	13 (20)	12 (16)
	<i>max</i>	82 (122)	82 (122)	80 (107)	122 (192)	104 (150)	101 (100)

calculated by bootstrapping. The power test is equal to 1 for any value of m , γ , k_m^0 , or theoretic test size α . The both test statistics (7) and (14) detect the change produced in the model.

The parameter γ does not modify the type I error probability. Concerning the change-point estimation precision, it does not improve in a significant way by the bootstrapping method or when the number m of historical data increases. This precision can be influenced by γ value when the test statistic (7), without bootstrapping, is used. It is worth mentioning that the obtained estimations of k_m^0 by the both methods are slightly biased, the delay time is of order $\approx +6$ observations, either for $m = 25$ or for $m = 100$ observations. Finally, if $\mathbf{f}(\mathbf{x}, \beta)$ is not bounded, the both test statistics detect the change-points, but the estimator bias of k_m^0 increases, if the change is 2 observations after m or 25 observations after m .

6. Proofs of the Theorems and Propositions

Here we present the proofs of the results stated in Sections 3 and 4.

Proof of Theorem 3.1

The proof follows the structure of the Theorem 2.1 proved by Horváth et al. (2004) for the linear case.

(i) Using Lemma 7.1 and Lemma 7.2 we have

$$\begin{aligned}
 \sup_{1 \leq k < \infty} \sum_{i=m+1}^{m+k} \hat{\varepsilon}_i / g(m, k, \gamma) &= \sup_{1 \leq k < \infty} \left(\sum_{i=m+1}^{m+k} \varepsilon_i - \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i \right) / g(m, k, \gamma) (1 + o_P(1)) \\
 &= \sigma \sup_{1 \leq k < \infty} [W_{1,m}(k) - \mathcal{D} \frac{k}{m} W_2(m)] / g(m, k, \gamma) (1 + o_P(1)).
 \end{aligned} \tag{16}$$

with $W_{1,m}$ and W_2 two independent Wiener processes on $[0, \infty)$. We obtain in a similar way as in the linear

case (Theorem 2.1 of Horváth et al., 2004)

$$\sup_{1 \leq k < \infty} \frac{|W_{1,m}(k) - \frac{k}{m} \mathcal{D}W_{2,m}(m)|}{g(m, k, \gamma)} \stackrel{\mathcal{L}}{=} \sup_{1 \leq k < \infty} \frac{|W_1(k) - \frac{k}{m} \mathcal{D}W_2(m)|}{g(m, k, \gamma)},$$

where $\{W_1(t)\}, \{W_2(t)\}$ are two independent Wiener processes on $[0, \infty)$. For all $K > 0$, by the continuity of $\{W_1(t) - \mathcal{D}tW_2(1)/(t/(1+t))^\gamma\}$ on $[0, K]$ we have

$$\max_{1 \leq k \leq mK} \frac{|W_1(k) - \frac{k}{m} \mathcal{D}W_2(m)|}{g(m, k, \gamma)} \stackrel{\mathcal{L}}{=} \max_{1 \leq k \leq mK} \frac{|W_1(\frac{k}{m}) - \frac{k}{m} \mathcal{D}W_2(1)|}{\left(1 + \frac{k}{m}\right)\left(\frac{k}{m+k}\right)^\gamma} \xrightarrow{a.s.} \sup_{0 \leq t \leq K} \frac{|W_1(t) - \mathcal{D}tW_2(1)|}{(1+t)\left(\frac{t}{1+t}\right)^\gamma}. \quad (17)$$

The relations (5.9) and (5.10) of Horváth et al.(2004) hold, then, for all $\delta > 0$,

$$\lim_{K \rightarrow \infty} \limsup_{m \rightarrow \infty} \mathbf{P} \left[\left| \sup_{mK \leq k < \infty} |W_1(\frac{k}{m}) - \frac{k}{m} \mathcal{D}W_2(1)| / (1 + \frac{k}{m}) \left(\frac{k}{m+k}\right)^\gamma - \mathcal{D}W_2(1) \right| > \delta \right] = 0,$$

$$\lim_{K \rightarrow \infty} \mathbf{P} \left[\left| \sup_{K < t < \infty} \frac{|W_1(t) - \mathcal{D}tW_2(1)|}{(1+t)\left(\frac{t}{1+t}\right)^\gamma} - \mathcal{D}W_2(1) \right| > \delta \right] = 0,$$

thus

$$\sup_{1 \leq k < \infty} \frac{|W_{1,m}(k) - \frac{k}{m} \mathcal{D}W_{2,m}(m)|}{g(m, k, \gamma)} \stackrel{\mathcal{L}}{\rightarrow} \sup_{0 \leq t < \infty} \frac{|W_1(t) - \mathcal{D}tW_2(1)|}{(1+t)\left(\frac{t}{1+t}\right)^\gamma}. \quad (18)$$

Let us consider the random processes $Z(t) = W_1(t) - \mathcal{D}tW_2(1)$ and $U(t) = (1 + \mathcal{D}^2 t)W\left(\frac{t}{1 + \mathcal{D}^2 t}\right)$, with $\{W(t), 0 \leq t < \infty\}$ a Wiener process. Their variances are $\text{Var}[Z(t)] = t + \mathcal{D}^2 t^2 = t(1 + \mathcal{D}^2 t)$, $\text{Var}[U(t)] = (1 + \mathcal{D}^2 t)^2 t / (1 + \mathcal{D}^2 t) = t(1 + \mathcal{D}^2 t)$. For $t_1 < t_2$, $\text{Cov}(Z(t_1), Z(t_2)) = \mathbf{E}[Z(t_1)Z(t_2)] + \mathcal{D}^2 t_1 t_2 = t_1 + \mathcal{D}^2 t_1 t_2 = t_1(1 + \mathcal{D}^2 t_2)$ and since $t/(1 + \mathcal{D}^2 t)$ is increasing in t , $\text{Cov}(U(t_1), U(t_2)) = (1 + \mathcal{D}^2 t_1)(1 + \mathcal{D}^2 t_2)t_1 / (1 + \mathcal{D}^2 t_1) = t_1(1 + \mathcal{D}^2 t_2)$. Thus, their variances and covariances coincide, we have $Z(t) \stackrel{\mathcal{L}}{=} U(t)$, for $0 \leq t < \infty$. Let us make the change of variable $t/(1 + \mathcal{D}^2 t) = y$, hence

$$\sup_{0 \leq t < \infty} \frac{|W_1(t) - tW_2(1)|}{(1+t)\left(\frac{t}{1+t}\right)^\gamma} \stackrel{\mathcal{L}}{=} \sup_{0 \leq y \leq \frac{1}{\mathcal{D}^2}} |W(y)| \frac{(1+y - \mathcal{D}^2 y)^\gamma}{y^\gamma}. \quad (19)$$

By the asymptotic properties of a nonlinear regression, we have that the variance error estimator $\hat{\sigma}_m^2$ is strongly converging to σ^2 , $|\hat{\sigma}_m - \sigma| = o_P(1)$. The assertion (i) follows by the last relation together the relations (16), (17), (18)-(19).

(ii) The proof is similar of (i). We give its outline:

$$\sup_{1 \leq k \leq T_m} \sum_{i=m+1}^{m+k} \hat{\varepsilon}_i / g(m, k, \gamma) = \sigma \sup_{1 \leq k \leq T_m} [W_{1,m}(k) - \mathcal{D} \frac{k}{m} W_2(m)] / g(m, k, \gamma) (1 + o_P(1))$$

$$\xrightarrow{a.s.} \sup_{m \rightarrow \infty} \sup_{0 \leq t \leq T} \frac{|W_1(t) - \mathcal{D}t W_2(1)|}{(1+t) \left(\frac{t}{1+t}\right)^\gamma} \stackrel{\mathcal{L}}{=} \sup_{0 \leq y \leq \frac{T}{1+\mathcal{D}^2 T}} |W(y)| \frac{(1+y - \mathcal{D}^2 y)^\gamma}{y^\gamma}.$$

■

Proof of Theorem 3.2

We choose this particular k : $\tilde{k}_m = k_m^0 + m$. We will prove that for this \tilde{k}_m we have $\lim_{m \rightarrow \infty} \left| \sum_{i=m+1}^{m+\tilde{k}_m} \hat{\varepsilon}_i \right| / g(m, \tilde{k}_m, \gamma) = \infty$. Let us consider the partial sum of the residuals after the first m observations

$$\sum_{i=m+1}^{m+\tilde{k}_m} \hat{\varepsilon}_i = \sum_{i=m+1}^{m+\tilde{k}_m} \varepsilon_i + \sum_{i=m+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m)] + \sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_i; \boldsymbol{\beta}^0)]. \quad (20)$$

Similar as for the Theorem 3.1 we have, for the first two terms of the right-hand side of (20),

$$\left| \sum_{i=m+1}^{m+\tilde{k}_m} [\varepsilon_i + f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m)] \right| / g(m, \tilde{k}_m, \gamma) = O_P(1) \quad (21)$$

and for the last term of the right-hand side of (20)

$$\begin{aligned} \sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_i; \boldsymbol{\beta}^0)] &= \sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)]] \\ &- \sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - E[f(\mathbf{X}; \boldsymbol{\beta}^0)]] + (\tilde{k}_m - k_m^0) (E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)] - E[f(\mathbf{X}; \boldsymbol{\beta}^0)]). \end{aligned} \quad (22)$$

Since $E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)] \neq E[f(\mathbf{X}; \boldsymbol{\beta}^0)]$, which implies, for the third term of the right-hand side of (22) that

$$\frac{(\tilde{k}_m - k_m^0) |E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)] - E[f(\mathbf{X}; \boldsymbol{\beta}^0)]|}{m^{1/2} g(m, \tilde{k}_m, \gamma)} = \frac{Cm}{m(1 + \frac{\tilde{k}_m}{m})(\frac{\tilde{k}_m/m}{1+\tilde{k}_m/m})^\gamma} > C > 0, \quad (23)$$

where C is a constant not depending of m . For the last relation, we have used that for $x > 1$ we have $\frac{1}{2} < \frac{x}{1+x} < 1$, then $(\frac{\tilde{k}_m/m}{1+\tilde{k}_m/m})^\gamma \in (2^{-\gamma}, 1)$ and $(1+x)^{-1} \geq 1$. On the other hand, using assumption (A5)

$$\sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)]] = \sum_{i=1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)]] - \sum_{i=1}^{m+k_m^0} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)]]$$

is of order $O_P(m + \tilde{k}_m)^{1/2} + O_P(m + k_m^0)^{1/2} = O_P(m + \tilde{k}_m)^{1/2}$. Moreover $m^{-1}(1 + \tilde{k}_m/m)^{-1}(m + \tilde{k}_m)^{1/2} \rightarrow 0$, as $m \rightarrow \infty$ and $(\frac{\tilde{k}_m/m}{1+\tilde{k}_m/m})^\gamma \in (2^{-\gamma}, 1)$. Thus, for the first term of the right-hand side of (20) we have

$$m^{-1/2} \sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - E[f(\mathbf{X}; \boldsymbol{\beta}_m^0)]] / g(m, \tilde{k}_m, \gamma) = o_P(1). \quad (24)$$

Similarly, for the second term of the right-hand side of (22)

$$m^{-1/2} \sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - E[f(\mathbf{X}; \boldsymbol{\beta}^0)]] / g(m, \tilde{k}_m, \gamma) = o_P(1). \quad (25)$$

Taking into account the relations (22)-(25) we can get, for the third term of the right-hand side of (20)

$$\liminf_{m \rightarrow \infty} m^{-1/2} \left| \sum_{i=m+k_m^0+1}^{m+\tilde{k}_m} [f(\mathbf{X}_i; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_i; \boldsymbol{\beta}^0)] \right| / g(m, \tilde{k}_m, \gamma) > 0. \quad (26)$$

The relations (20), (21), (26) imply that we found one \tilde{k}_m such that $\lim_{m \rightarrow \infty} m^{-1/2} \left| \sum_{i=m+1}^{m+\tilde{k}_m} \hat{\varepsilon}_i \right| / g(m, \tilde{k}_m, \gamma) > 0$. Thus $\lim_{m \rightarrow \infty} \left| \sum_{i=m+1}^{m+\tilde{k}_m} \hat{\varepsilon}_i \right| / g(m, \tilde{k}_m, \gamma) = \infty$. Then $\lim_{m \rightarrow \infty} \sup_{1 \leq k < \infty} \left| \sum_{i=m+1}^{m+k} \hat{\varepsilon}_i \right| / g(m, k, \gamma) = \infty$. The theorem follows. ■

Proof of Proposition 4.1

It is clear that $|Z_{k,n} - \mu_{k,n}| \leq |Z_{k,n} - \mu_{k,n}| \leq \max_k |Z_{k,n} - \mu_{k,n}|$. Then $\mathbf{P}[\max_k |Z_{k,n} - \mu_{k,n}| \geq \epsilon] \geq \mathbf{P}[|Z_{k,n}| - |\mu_{k,n}| \geq \epsilon] = \mathbf{P}[|Z_{k,n}| \geq \epsilon + |\mu_{k,n}|] \geq \mathbf{P}[\max_k (|Z_{k,n}|) \geq \epsilon + |\mu_{k,n}|] \geq \mathbf{P}[\max_k (|Z_{k,n}|) \geq 2\epsilon]$. For the last inequality we have used: for all $\epsilon > 0$ there exists a natural number n_ϵ such that for all $n \geq n_\epsilon$ we have $|\mu_{k,n}| \leq \epsilon$. ■

Proof of Proposition 4.2

We denote by $e_2 \equiv -l/m \sum_{j=1}^m \varepsilon \mathcal{U}_{m,k(j)}$ and we remind the notation $D_A \equiv \mathbf{A}' \mathbf{A}$. Without loss of generality, we take $l \leq k$, the other cases are similar. Consider now the following random variable, for $i = m+1, \dots, m+l$,

$$G_{i,k} \equiv D_A^{-1} \frac{1}{m} \left(\sum_{j=1}^m \mathbf{f}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon \mathcal{U}_{m,k(j)} \right) \mathbf{f}(\mathbf{X}_i; \boldsymbol{\beta}^0) - \frac{1}{m} \sum_{j=1}^m \varepsilon \mathcal{U}_{m,k(j)}.$$

Consequently, $-I_2 + e_2 = \sum_{i=m+1}^{m+l} G_{i,k}$. The conditional expectation of $-I_2 + e_2$ is

$$\begin{aligned} \mathbf{E}_{k,m}^* [-I_2 + e_2] &= \frac{1}{m} \sum_{j=1}^m \mathbf{f}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \frac{1}{m+k} \sum_{i=1}^{m+k} \varepsilon_i \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l) - \frac{l}{m} \sum_{j=1}^m \frac{1}{m+k} \sum_{i=1}^{m+k} \varepsilon_i \\ &= \bar{\varepsilon}_{m+k} \left\{ D_A^{-1} \left[\frac{m+l}{m} \sum_{j=1}^m \mathbf{f}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \frac{1}{m+l} \sum_{i=1}^{m+l} \mathbf{f}(\mathbf{X}_i; \boldsymbol{\beta}^0) - \frac{m}{m} \sum_{j=1}^m \mathbf{f}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \frac{1}{m} \sum_{i=1}^m \mathbf{f}(\mathbf{X}_i; \boldsymbol{\beta}^0) \right] - l \right\}. \end{aligned}$$

On the other hand, by assumption (A1) for all $\epsilon > 0$, there exists $M_1 > 0$ such that $\mathbf{P}[m^{1/2} |\bar{\varepsilon}_{m+k}| > M_1] < \epsilon$.

Thus, taking also into account the assumption (A4) for $\mathbf{f}(\mathbf{X}_i; \boldsymbol{\beta}^0)$, we get

$$\mathbf{E}_{k,m}^* \left[\frac{-I_2 + e_2}{g(m, l, \gamma)} \right] = \frac{o_P(m+l) \bar{\varepsilon}_{m+k}}{m^{1/2} \left(1 + \frac{l}{m}\right) \left(\frac{l/m}{1+l/m}\right)^\gamma} = \frac{o_P(1 + l/m) m^{1/2} \bar{\varepsilon}_{m+k}}{\left(1 + \frac{l}{m}\right) \left(\frac{l/m}{1+l/m}\right)^\gamma}.$$

Using the relation (5), the last relation is $o_P(1)m^{1/2}\bar{\varepsilon}_{m+k} = o_P(1)O_P(1) = o_P(1)$, for all $l, k = 1, \dots, T_m$. Consequently

$$\mathbf{E}_{k,m}^* \left[\frac{-\sum_{i=m+1}^{m+l} G_{i,k}}{g(m, l, \gamma)} \right] = o_P(1).$$

for all $l, k = 1, \dots, T_m$. Then, we are in the conditions to apply the inequality (12) for the random variable $G_{i,k}$ and the sequence $(b_l \equiv g(m, l, \gamma))$. Hence, for any $\epsilon > 0$, there exists a natural number m_ϵ such that for $m \geq m_\epsilon$,

$$\mathbf{P}_{k,m}^* \left[\sup_{1 \leq l \leq T_m} \frac{|\sum_{i=m+1}^{m+l} G_{i,k}|}{g(m, l, \gamma)} \geq 2\epsilon \right] \leq \frac{1}{\epsilon^2} \sum_{l=1}^{T_m} \frac{\mathbf{E}_{k,m}^*[G_{i,k}^2]}{g^2(m, l, \gamma)}. \quad (27)$$

By elementary algebra, using the fact that for $j \neq j'$, $\mathbf{E}_{k,m}^*[\varepsilon u_{m,k(j)} \cdot \varepsilon u_{m,k(j')}] = \mathbf{E}_{k,m}^*[\varepsilon u_{m,k(j)}] \cdot \mathbf{E}_{k,m}^*[\varepsilon u_{m,k(j')}] = (m+k)^{-2} \left(\sum_{a=1}^{m+k} \varepsilon_a \right)^2$, yield

$$\begin{aligned} \mathbf{E}_{k,m}^*[G_{i,k}^2] &= (\bar{\varepsilon}_{m+k})^2 \left[m^{-1} \sum_{j=1}^m \left(D_A^{-1} \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) - 1 \right) \right]^2 \\ &\quad + \left(\bar{\varepsilon}_{m+k}^2 - (\bar{\varepsilon}_{m+k})^2 \right) \left[m^{-2} \sum_{j=1}^m \left(D_A^{-1} \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) - 1 \right)^2 \right]. \end{aligned} \quad (28)$$

By the relation (30) of Hušková and Kirch (2012), we get, for a constant $C_1 > 0$: $g(m, l, \gamma) \geq C_1(m^{1/2-\gamma} l^\gamma \mathbb{1}_{l \leq m} + m^{-1/2} l \mathbb{1}_{l > m})$. Thus, for $l > m$ we have $g^{-2}(m, l, \gamma) < m C_1^{-2} l^{-2} < C_1^{-2} m^{-1} \rightarrow 0$, as $m \rightarrow \infty$ and for $1 \leq l \leq m$, $g^{-2}(m, l, \gamma) \leq C_1^{-2} m^{-1+2\gamma} l^{-2\gamma} \leq C_1^{-2} m^{-1+2\gamma} \rightarrow 0$, as $m \rightarrow \infty$. Under the assumptions (A4) and (A6) we have the following inequalities with a probability close to 1

$$\frac{1}{m^2} \sum_{l=1}^m \left[\sum_{j=1}^m \left(D_A^{-1} \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) - 1 \right) \right]^2 \frac{1}{g^2(m, l, \gamma)} \leq C \sum_{l=1}^m \frac{1}{g^2(m, l, \gamma)} \leq C.$$

On the other hand, by the Cauchy-Schwarz inequality we readily have with a probability 1

$$\begin{aligned} &\frac{1}{m^2} \sum_{l=m+1}^{T_m} \left[\sum_{j=1}^m \left(\dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) D_A^{-1} - 1 \right) \right]^2 \frac{1}{g^2(m, l, \gamma)} \\ &\leq \left\{ \frac{1}{m^4} \sum_{l=m+1}^{T_m} \left[\sum_{j=1}^m \left(\dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) D_A^{-1} - 1 \right) \right]^4 \right\}^{1/2} \left\{ \sum_{l=m+1}^{T_m} \frac{1}{g^4(m, l, \gamma)} \right\}^{1/2} \\ &\leq \left\{ C(T_m - m) \sum_{l=m+1}^{T_m} \frac{1}{g^4(m, l, \gamma)} \right\}^{1/2} \leq \left\{ C(T_m - m) m^2 \sum_{l=m+1}^{T_m} \frac{1}{l^4} \right\}^{1/2} = C \left\{ m^2(T_m - m) \left(\frac{1}{m^4} - \frac{1}{T_m^4} \right) \right\}^{1/2} = C. \end{aligned}$$

were used that $m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0)$, $m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0)$, $T_m^{-1} \sum_{j=1}^{T_m} \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0)$ are converging by assumption (A4). Hence

$$\frac{1}{m^2} \sum_{l=1}^{T_m} \left[\sum_{j=1}^m \left(D_A^{-1} \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) - 1 \right) \right]^2 \frac{1}{g^2(m, l, \gamma)} = O_P(1). \quad (29)$$

For the second term of the right-hand side of the relation (28) we have: $\dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0)$
 $= \text{trace}(\dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0)) = \dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0)$. Consequently, since for $l \leq m$, $g^{-2}(m, l, \gamma) \leq C m^{-1+2\gamma}$, we have with a probability close to 1 that $m^{-2} \sum_{l=1}^m \sum_{j=1}^m \{\dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) D_A^{-1} - 1\}^2 g^{-2}(m, l, \gamma)$ is less than or equal to $2m^{-1} \sum_{l=1}^m \left[\dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \mathbf{B}_m \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) D_A^{-2} + 1 \right] g^{-2}(m, l, \gamma) \leq C m^{-1+2\gamma} m^{-1} \cdot \sum_{l=1}^m [\dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \mathbf{B}_m \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) D_A^{-2} + 1] \leq C m^{-1+2\gamma} m^{-1} \sum_{l=1}^m [\|\dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0)\|_2^2 \|\mathbf{B}_m\|_2 + 1] = C m^{-1+2\gamma} \rightarrow 0$, for $m \rightarrow \infty$. For the last relation we used the assumption (A4).

We have in the other hand $m^{-2} \sum_{l=m+1}^{T_m} \sum_{j=1}^m \left(\dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) D_A^{-1} - 1 \right)^2 g^{-2}(m, l, \gamma)$ is less than or equal to $2m^{-1} \sum_{l=m+1}^{T_m} g^{-2}(m, l, \gamma) m^{-1} \sum_{j=1}^m \left[D_A^{-2} \dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) + 1 \right]$
 $= 2m^{-1} \sum_{l=m+1}^{T_m} g^{-2}(m, l, \gamma) [D_A^{-2} \dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \mathbf{B}_m \dot{\mathbf{f}}(\mathbf{X}_l; \boldsymbol{\beta}^0) + 1]$ and by the Cauchy-Schwarz inequality
 $\leq 2m^{-1} \left\{ \sum_{l=m+1}^{T_m} [\|\dot{\mathbf{f}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0)\|_2^2 \|\mathbf{B}_m\|_2 + 1] \right\}^{1/2} \left\{ \sum_{l=m+1}^{T_m} g^{-4}(m, l, \gamma) \right\}^{1/2}$
 $\leq C m^{-1} \left\{ (T_m - m) \sum_{l=m+1}^{T_m} g^{-4}(m, l, \gamma) \right\}^{1/2} \leq C m^{-1} \rightarrow 0, \quad \text{for } m \rightarrow \infty, \quad (30)$

uniformly in k . Since $\bar{\varepsilon}_{m+k} \xrightarrow{a.s.} 0$ and $\bar{\varepsilon}_{m+k}^2 - (\bar{\varepsilon}_{m+k})^2 = \hat{\sigma}_{m,k}^2 \xrightarrow{a.s.} \sigma^2$, and using the results (28), (29), (30), we obtain by (27) that $\sup_{1 \leq k < \infty} \mathbf{P}_{k,m}^* \left[\sup_{1 \leq l \leq T_m} \left| \sum_{i=m+1}^{m+l} G_i \right| / g(m, l, \gamma) \geq \epsilon \right] \xrightarrow{P} 0$. ■

Proof of Proposition 4.3

Using the Proposition 4.2 and the Lemmas 7.3, 7.4, the proof is similar to that of the Lemma 5 of Hušková and Kirch (2012). ■

Proof of Theorem 4.1

Using the Proposition 4.3, the Theorems 3.1, 3.2 and Lemma 7.5, the proof is similar to that of the Theorem 1 of Horváth and Kirch (2012). ■

7. Appendix

In this section useful Lemmas to prove the main results of Sections 3 and 4 are given. We recall the notations: $\mathbf{A} \equiv E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)]$, $\mathbf{B} \equiv E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)^t]$, $A_i \equiv \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \mathbf{B}^{-1} \mathbf{A}$.

7.1. Lemmas for Section 3

Lemma 7.1 Suppose that assumptions (A1)-(A4) hold. Under the hypothesis H_0 we have, as $m \rightarrow \infty$,

$$\sup_{1 \leq k < \infty} \left| \sum_{i=m+1}^{m+k} \hat{\varepsilon}_i - \left(\sum_{i=m+1}^{m+k} \varepsilon_i - \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i \right) \right| / g(m, k, \gamma) = o_P(1).$$

Proof of Lemma 7.1

Under the hypothesis H_0 , $\sum_{i=m+1}^{m+k} \hat{\varepsilon}_i = \sum_{i=m+1}^{m+k} \varepsilon_i - \sum_{i=m+1}^{m+k} [f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m) - f(\mathbf{X}_i; \boldsymbol{\beta}^0)]$. Then, by a Taylor expansion of $f(\mathbf{X}_i; \boldsymbol{\beta})$ in a neighborhood of $\boldsymbol{\beta}^0$

$$\sum_{i=m+1}^{m+k} \hat{\varepsilon}_i - \left(\sum_{i=m+1}^{m+k} \varepsilon_i - \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i \right) = \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i - \sum_{i=m+1}^{m+k} [f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m) - f(\mathbf{X}_i; \boldsymbol{\beta}^0)] = \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i - (\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0)^t \sum_{i=m+1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_i; \tilde{\boldsymbol{\beta}}_m), \quad (31)$$

with $\tilde{\boldsymbol{\beta}}_m = \boldsymbol{\beta}^0 + \theta(\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0)$, $\theta \in [0, 1]^q$. We know that the LS estimator $\hat{\boldsymbol{\beta}}_m$ of parameter in a linear model is \sqrt{m} -consistent (see Seber and Wild, 2003) $\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0 = O_P(m^{-1/2})$. On the other hand, by the triangle inequality

$$\left\| \sum_{i=m+1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_i; \tilde{\boldsymbol{\beta}}_m) - k E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)] \right\|_1 \leq \left\| \sum_{i=1}^{m+k} (\dot{\mathbf{f}}(\mathbf{X}_i; \tilde{\boldsymbol{\beta}}_m) - E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)]) \right\|_1 + \left\| \sum_{i=1}^m (\dot{\mathbf{f}}(\mathbf{X}_i; \tilde{\boldsymbol{\beta}}_m) - E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)]) \right\|_1 \quad (32)$$

Generally, for any $n \geq m$ and $\boldsymbol{\beta}$ in a $m^{-1/2}$ -neighborhood of $\boldsymbol{\beta}^0$, by assumption (A4) for $\dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)$, using the law of iterated logarithm we have that for all $\epsilon > 0$, there exists a $M_2 > 0$ such that $P[n^{-1/2} \sum_{i=1}^n \|\dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) - \mathbf{A}\|_1 \geq M_2] \leq \epsilon$. Together with assumption (A2), it holds that, for all $\boldsymbol{\beta}$ in a $m^{-1/2}$ -neighborhood of $\boldsymbol{\beta}^0$

$$\begin{aligned} & \left\| \sum_{i=1}^n [\dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}) - E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)]] \right\|_1 \leq \left\| \sum_{i=1}^n [\dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}) - \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)] \right\|_1 + \left\| \sum_{i=1}^n [\dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) - E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)]] \right\|_1 \\ & = \|\boldsymbol{\beta} - \boldsymbol{\beta}^0\|_1 O_P(n) + O_P(n^{1/2}) = O_P(n \cdot m^{-1/2}) + O_P(n^{1/2}) \text{ uniformly in } \boldsymbol{\beta}. \end{aligned}$$

Thus, the right-hand side of (32) becomes $(m+k)m^{-1/2} + (m+k)^{1/2} + m^{1/2} + m^{1/2} = (m+k)m^{-1/2} + (m+k)^{1/2} + 2m^{1/2}$. Hence, for the last term of (31) we have

$$\sup_{1 \leq k < \infty} \frac{|\langle \hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0 \rangle^t \sum_{i=m+1}^{m+k} (\dot{\mathbf{f}}(\mathbf{X}_i; \tilde{\boldsymbol{\beta}}_m) - E[\dot{\mathbf{f}}(\mathbf{X}; \boldsymbol{\beta}^0)])|}{g(m, k, \gamma)} = O_P(m^{-1/2}) \sup_{1 \leq k < \infty} \frac{m^{1/2} + (m+k)m^{-1/2} + (m+k)^{1/2}}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m+k}\right)^\gamma}$$

$$= C \sup_{1 \leq k < \infty} \frac{1 + \left(1 + \frac{k}{m}\right) + \left(1 + \frac{k}{m}\right)^{1/2}}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m+k}\right)^\gamma} \leq C \sup_{1 \leq k < \infty} \frac{2 + \frac{k}{m}}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m} \cdot \frac{1}{1 + \frac{k}{m}}\right)^\gamma}.$$

Since, for $k \leq m$, $\frac{k}{k+m} > \frac{1}{2m}$ and for $x > 0$ we have $(1+x)^{-1} < 1$, we can write

$$\sup_{1 \leq k \leq m} \frac{2 + \frac{k}{m}}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m} \cdot \frac{1}{1 + \frac{k}{m}}\right)^\gamma} \leq \frac{3}{m^{1/2} \left(\frac{1}{2m}\right)^\gamma} = 3 \cdot 2^{\gamma-1} m^{-1/2+\gamma} = o(1).$$

For all $x \geq 1$, we have that $\left(\frac{x+1}{x}\right)^\gamma \leq 2^\gamma$ and $(1+x)^{-1} \leq 2^{-1}$. Then

$$\sup_{m \leq k < \infty} \frac{2 + \frac{k}{m}}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m} \cdot \frac{1}{1 + \frac{k}{m}}\right)^\gamma} \leq \frac{1}{m^{1/2}} \left(\frac{2^\gamma}{2} + 2^\gamma\right) = o(1).$$

Hence, (31) becomes

$$\sum_{i=m+1}^{m+k} \hat{\varepsilon}_i - \left(\sum_{i=m+1}^{m+k} \varepsilon_i - \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i \right) = \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i + k(\hat{\beta}_m - \beta^0)^t E[\dot{\mathbf{f}}(\mathbf{X}; \beta^0)](1 + o_P(1)). \quad (33)$$

On the other hand, $\hat{\beta}_m$ is the least squares estimator of β^0 , calculated for $i = 1, \dots, m$,

$$\begin{aligned} 0 &= \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_m) [Y_i - f(\mathbf{X}_i; \hat{\beta}_m)] = \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \hat{\beta}_m) [\varepsilon_i - (\hat{\beta}_m - \beta^0)^t \dot{\mathbf{f}}(\mathbf{X}_i; \tilde{\beta}_m)] \\ &= \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \varepsilon_i + \sum_{i=1}^m \ddot{\mathbf{f}}(\mathbf{X}_i; \tilde{\beta}_m) (\hat{\beta}_m - \beta^0)^t \varepsilon_i - \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \dot{\mathbf{f}}^t(\mathbf{X}_i; \beta^0) (\hat{\beta}_m - \beta^0) \\ &\quad - 1/2 (\hat{\beta}_m - \beta^0)^t \sum_{i=1}^m \ddot{\mathbf{f}}(\mathbf{X}_i; \tilde{\beta}_m) \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) (\hat{\beta}_m - \beta^0). \end{aligned}$$

Using the assumptions (A1), (A2) and the Cauchy-Schwarz inequality, we obtain

$$0 = \left[\sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \varepsilon_i - \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \dot{\mathbf{f}}^t(\mathbf{X}_i; \beta^0) (\hat{\beta}_m - \beta^0) \right] (1 + o_P(1)). \quad (34)$$

Then, by relation (34) below

$$\hat{\beta}_m - \beta^0 = \mathbf{B}_m^{-1} \left(\frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \varepsilon_i \right) (1 + o_P(1)) = \mathbf{B}^{-1} \left(\frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \varepsilon_i \right) (1 + o_P(1)), \quad (35)$$

again too

$$k(\hat{\beta}_m - \beta^0)^t E[\dot{\mathbf{f}}(\mathbf{X}_i; \beta^0)] = \left(\frac{k}{m} \sum_{i=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_i; \beta^0) \varepsilon_i \right) \mathbf{B}^{-1} E[\dot{\mathbf{f}}(\mathbf{X}; \beta^0)] (1 + o_P(1)) = \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i (1 + o_P(1))$$

and we replace next in (33). To complete the proof, we must prove that for (34) that $\sup_{1 \leq k < \infty} k/(mg(m, k, \gamma))o_P(1) = o_P(1)$. Using (A1)-(A4) and the fact that $\hat{\beta}_m - \beta^0 = O_P(m^{-1/2})$ we deduce that

$$\sup_{1 \leq k < \infty} \frac{\|(\hat{\beta}_m - \beta^0)^t \frac{k}{m} \sum_{i=1}^m \ddot{\mathbf{f}}(\mathbf{X}_i; \tilde{\beta}_m) \varepsilon_i\|_1}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m+k}\right)^\gamma} \leq K_m \frac{1}{m} \left\| \sum_{i=1}^m \ddot{\mathbf{f}}(\mathbf{X}_i; \tilde{\beta}_m) \varepsilon_i \right\|_1 = o_P(1),$$

with K_m given by (5). Similarly

$$\sup_{1 \leq k < \infty} \frac{\|(\hat{\beta}_m - \beta^0)^t \frac{k}{m} \sum_{i=1}^m \{\dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \dot{\mathbf{f}}^t(\mathbf{X}_i; \beta^0) - E[\dot{\mathbf{f}}(\mathbf{X}; \beta^0) \dot{\mathbf{f}}^t(\mathbf{X}; \beta^0)]\|_1}{g(m, k, \gamma)} = K_m o_P(1) = o_P(1).$$

Using (5), with the Cauchy-Schwarz inequality for matrix, we have

$$\begin{aligned} & \sup_{1 \leq k < \infty} \frac{\|(\hat{\beta}_m - \beta^0)^t \sum_{i=1}^m \ddot{\mathbf{f}}(\mathbf{X}_i; \tilde{\beta}_m) \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) (\hat{\beta}_m - \beta^0)^t\|_2}{g(m, k, \gamma)} \\ & \leq \sup_{1 \leq k < \infty} K_m \left\| \frac{k}{m} \sum_{i=1}^m \ddot{\mathbf{f}}(\mathbf{X}_i; \tilde{\beta}_m) \dot{\mathbf{f}}(\mathbf{X}_i; \beta^0) \right\|_1 \cdot \|\hat{\beta}_m - \beta^0\|_1 \cdot O_P(\|\hat{\beta}_m - \beta^0\|_1) = o_P(1). \end{aligned}$$

■

Lemma 7.2 Suppose that assumptions (A1)-(A3) hold. Under the hypothesis H_0 , there exists two independent Wiener processes $\{W_{1,m}(t), 0 \leq t < \infty\}$ and $\{W_{2,m}(t), 0 \leq t < \infty\}$ such that, for $m \rightarrow \infty$,

$$\sup_{1 \leq k < \infty} \left| \left(\sum_{i=m+1}^{m+k} \varepsilon_i - \frac{k}{m} \sum_{i=1}^m A_i \varepsilon_i \right) - \left(\sigma W_{1,m}(k) - \frac{k}{m} \sigma \mathcal{D} W_{2,m}(m) \right) \right| / g(m, k, \gamma) = o_P(1).$$

Proof of Lemma 7.2

The random variables $\{\sum_{i=m+1}^{m+k} \varepsilon_i, 1 \leq k < \infty\}$ and $\{\sum_{i=1}^m A_i \varepsilon_i\}$ are independent. It is obvious that, since \mathbf{X}_i is independent of ε_i , we have $E[A_i \varepsilon_i] = 0$, $Var[A_i \varepsilon_i] = E[A_i^2] E[\varepsilon_i^2]$. On the other hand, $E[\varepsilon_i^2] = \sigma^2$ and $E[A_i^2] = \mathbf{A}^t \mathbf{B}^{-1} \mathbf{A}$. By an argument similar to the one used in Horváth et al.(2004), Lemma 5.3., we obtain $\sup_{1 \leq k < \infty} |\sum_{i=m+1}^{m+k} \varepsilon_i - \sigma W_{1,m}(k)| / k^{1/\gamma} = o_P(1)$ and $\sum_{i=1}^m A_i \varepsilon_i - \sigma \mathcal{D} W_{2,m}(m) = o_P(m^{1/\nu})$, as $m \rightarrow \infty$, $\nu > 2$. The rest of proof is similar to that of the Lemme 5.3. of Horváth et al.(2004). ■

7.2. Lemmas for Section 4

We recall that (see the decomposition of $g\tilde{\Gamma}$ given in Section 4): $g(m, l, \gamma)\tilde{\Gamma}(m, l, \gamma)(\varepsilon_{m,k}^*(1), \dots, \varepsilon_{m,k}^*(m+l)) = I_1 + I_2 + \mathcal{R}_m$. More precisely, \mathcal{R}_m have the decomposition: $\mathcal{R}_m \equiv \sum_{j=3}^8 I_j$, with

$$I_3 \equiv \sum_{i=m+1}^{m+l} \mathbb{1}_{k \leq k_m^0} \mathbb{1}_{u_{m,k}(i) < m+k_m^0} \left[f(\mathbf{X}_{u_{m,k}(i)}; \beta^0) - f(\mathbf{X}_{u_{m,k}(i)}; \hat{\beta}_{m+k}) \right],$$

$$\begin{aligned}
I_4 &\equiv \sum_{i=m+1}^{m+l} \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(i) < m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k}) \right], \\
I_5 &\equiv \sum_{i=m+1}^{m+l} \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(i) > m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k}) \right], \\
I_6 &\equiv - \left(\frac{1}{m} \sum_{j=1}^m \mathbf{f}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbb{1}_{k \leq k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(j) < m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \hat{\boldsymbol{\beta}}_{m+k}) \right] \right) \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l), \\
I_7 &\equiv - \left(\frac{1}{m} \sum_{j=1}^m \mathbf{f}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(j) < m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \hat{\boldsymbol{\beta}}_{m+k}) \right] \right) \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l), \\
I_8 &\equiv - \left(\frac{1}{m} \sum_{j=1}^m \mathbf{f}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(j) > m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \hat{\boldsymbol{\beta}}_{m+k}) \right] \right) \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l).
\end{aligned}$$

Under the hypothesis H_0 , $I_4 = I_5 = I_7 = I_8 = 0$. Let us consider now

$$\tilde{I}_3 \equiv I_3 - \frac{l}{m+k} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})], \quad \tilde{I}_6 \equiv I_6 + \frac{l}{m+k} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})].$$

Lemma 7.3 *Under the assumptions (A1)-(A4), for all $\epsilon > 0$, we have*

$$\mathbf{P}_{m,k}^* \left[\max_{1 \leq l \leq T_m} \frac{|\tilde{I}_3|}{g(m, l, \gamma)} \geq \epsilon \right] \rightarrow 0 \quad \text{in probability, uniformly in } k, \quad \text{as } m \rightarrow \infty, \quad (36)$$

$$\sup_{1 \leq k < \infty} \mathbf{P}_{m,k}^* \left[\max_{1 \leq l \leq T_m} \frac{|\tilde{I}_6|}{g(m, l, \gamma)} \geq \epsilon \right] \rightarrow 0, \quad \text{in probability, as } m \rightarrow \infty, \quad (37)$$

whether under H_0 or H_1 .

Proof of Lemma 7.3 Let us consider the random variable, for $i = m+1, \dots, m+l$,

$$\mathcal{Z}_{k,m}(i) = f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k}) - \frac{1}{m+k} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})]. \quad (38)$$

Then $\mathbf{E}_{m,k}^*[\mathcal{Z}_{k,m}(i)] = (m+k)^{-1} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})] - (m+k)^{-1} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})] = 0$.

Hence, since $\tilde{I}_3 = \sum_{i=m+1}^{m+l} \mathcal{Z}_{k,m}(i)$, we have $\mathbf{E}_{m,k}^*[\tilde{I}_3] = 0$.

For \tilde{I}_6 let us consider $l \leq k$, for two other cases the arguments are like. Since \tilde{I}_6 is scalar, using (A4), from an equality to an other we apply the *trace* operator, the conditional expectation $\mathbf{E}_{m,k}^*[-I_6]$ is equal to

$$\frac{1}{m+k} \sum_{i=1}^{m+k} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_{m+k})] \left(\frac{1}{m} \sum_{j=1}^m \mathbf{f}(\mathbf{X}_j; \boldsymbol{\beta}^0) \right) \mathbf{B}_{m+k}^{-1} \mathbf{B}_{m+k} D_A^{-1} \left(\sum_{i=m+1}^{m+l} \mathbf{f}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) \right)$$

$$\begin{aligned}
&= \frac{D_A^{-1}}{m+k} \sum_{i=1}^{m+k} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_{m+k})] \left[\frac{m+l}{m+l} \sum_{i=1}^{m+l} \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) - \frac{m}{m} \sum_{i=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) \right] \left(\frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \right) \\
&= [(m+l)\mathbf{A}^t - m\mathbf{A}^t] D_A^{-1} \mathbf{A} (1 + o_P(1)) \frac{1}{m+k} \sum_{i=1}^{m+k} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_{m+k})] \\
&= l(1 + o_P(1)) \frac{1}{m+k} \sum_{i=1}^{m+k} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_{m+k})].
\end{aligned}$$

Hence, $\mathbf{E}_{m,k}^* [\tilde{I}_6] = o_P(1) \frac{l}{m+k} \sum_{i=1}^{m+k} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_{m+k})] = l(m+k)^{-1/2} o_P(1) = lm^{-1/2} o_P(1)$. On the other hand, the conditional variance of $\mathcal{Z}_{k,m}(i)$ is

$$\begin{aligned}
\text{Var}_{m,k}^* [\mathcal{Z}_{k,m}(i)] &= \mathbf{E}^* [f(\mathbf{X}_{u_{m,k}(i)}; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_{u_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k})]^2 = \frac{1}{m+k} \sum_{i=1}^{m+k} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_{m+k})]^2 \\
&\leq \frac{\|\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0\|_2^2}{m+k} \left[\sum_{i=1}^{m+k} \|\dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)\|_2^2 \right] (1 + o_P(1)) = C(m+k)^{-1} (1 + o_P(1)),
\end{aligned}$$

which is $o_P(1)$ uniformly in k . We apply the Hájek-Rényi inequality for $\mathcal{Z}_{k,m}(i)$: for all $\epsilon > 0$,

$$\mathbf{P}_{m,k}^* \left[\max_{1 \leq l \leq T_m} \frac{1}{g(m, l, \gamma)} \left| \sum_{i=m+1}^{m+l} \mathcal{Z}_{k,m}(i) \right| \geq \epsilon \right] \leq \frac{1}{\epsilon^2} \sum_{l=1}^{T_m} \frac{\mathbf{E}_{m,k}^* [\mathcal{Z}_{k,m}^2(l)]}{g^2(m, l, \gamma)} \leq \frac{1}{\epsilon^2} \frac{C}{m+k} \sum_{l=1}^{T_m} \frac{1}{g^2(m, l, \gamma)} = \frac{C}{\epsilon^2(m+k)},$$

and the relation (36) follows.

For \tilde{I}_6 , we can write, $\tilde{I}_6 = -\sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathcal{Z}_{k,m}(j) \mathbf{B}_m^{-1} \mathbf{c}_1(m, k, l)$. Then, since

$$\frac{\mathbf{E}_{m,k}^* [\tilde{I}_6]}{g(m, l, \gamma)} = \frac{l/m}{(1 + l/m) \left(\frac{l}{l+m} \right)^\gamma} o_P(1) = \left(\frac{l/m}{1 + l/m} \right)^{1-\gamma} o_P(1) \quad \text{and} \quad \left(\frac{l/m}{1 + l/m} \right)^{1-\gamma} \quad \text{is bounded by the relation (5),}$$

combined with $\sup_{1 \leq k < \infty} \sup_{1 \leq l \leq T_m} \|\mathbf{c}_1(m, k, l)\|_2 / m \leq C$ and since the random trial of bootstrap are independently, then $\mathcal{Z}_{k,m}(j)$ are also independently, we have $\text{Var}_{m,k}^* [\tilde{I}_6] = m^{-1} \text{Var}_{m,k}^* [\mathcal{Z}_{k,m}(j)] \mathbf{c}_1^t(m, k, l) \mathbf{B}_{m+k}^{-1} \mathbf{c}_1(m, k, l) = o_P(1)$. Hence, by the Bienaymé-Tchebychev inequality, Proposition 4.1 and inequality (12), the relation (37) follows. \blacksquare

Let be (the expressions of I_4, I_5, I_7, I_8 are given before the Lemma 7.3):

$$\begin{aligned}
\tilde{I}_4 &\equiv I_4 - \mathbb{1}_{k > k_m^0} \frac{l}{m+k} \sum_{j=1}^{m+k_m^0} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})], & \tilde{I}_5 &= I_5 - \mathbb{1}_{k > k_m^0} \frac{l}{m+k} \sum_{j=m+k_m^0+1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})], \\
\tilde{I}_7 &\equiv I_7 + \mathbb{1}_{k > k_m^0} \frac{l}{m+k} \sum_{j=1}^{m+k_m^0} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})], & \tilde{I}_8 &= I_8 + \mathbb{1}_{k > k_m^0} \frac{l}{m+k} \sum_{j=m+k_m^0+1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})].
\end{aligned}$$

Lemma 7.4 *Under the assumptions (A1)-(A5), if the hypothesis H_1 is true, we have that for all $\epsilon > 0$ there exists $M > 0$ such that we have in probability*

$$\sup_{1 \leq k < \infty} \mathbf{P}_{m,k}^* \left[\max_{1 \leq l \leq T_m} \frac{|\tilde{I}_j|}{g(m, l, \gamma)} \geq M \right] \leq \epsilon + o_{\mathbf{P}}(1), \quad \text{for } j \in \{4, 5, 7, 8\}. \quad (39)$$

Proof of Lemma 7.4

For \tilde{I}_4 . Let be consider the following random variable

$$\tilde{Z}_{m,k}(i) \equiv \mathbb{1}_{\mathcal{U}_{m,k}(i) < m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k}) \right] - \frac{1}{m+k} \sum_{i=1}^{m+k_m^0} [f(\mathbf{X}_i; \boldsymbol{\beta}^0) - f(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_{m+k})].$$

It is obvious that $\mathbf{E}_{m,k}^*[\tilde{Z}_{m,k}(i)] = 0$. Since $\tilde{I}_4 = \sum_{i=m+1}^{m+l} \tilde{Z}_{m,k}(i)$, we have that $\mathbf{E}_{m,k}^*[\tilde{I}_4] = 0$. For the conditional variance of $\tilde{Z}_{m,k}(i)$ we have, using a quadratic Taylor expansion and the triangular inequality

$$\begin{aligned} \text{Var}_{m,k}^*[\tilde{Z}_{m,k}(i)] &= \mathbf{E}^* \left[\mathbb{1}_{\mathcal{U}_{m,k}(i) < m+k_m^0} [f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k})]^2 \right] = \frac{1}{m+k} \sum_{j=1}^{m+k_m^0} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})]^2 \\ &= \frac{1}{m+k} \sum_{j=1}^{m+k_m^0} \left[(\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0)^t \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) + \frac{(\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0)^t}{2} \ddot{\mathbf{f}}(\mathbf{X}_j; \tilde{\boldsymbol{\beta}}_{m,k}) (\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0) \right]^2 \\ &\leq \frac{(m+k_m^0) \|\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0\|_2^2}{m+k} \|\mathbf{B}_{m+k_m^0}\|_2^2 + \frac{\|\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0\|_2^2}{m+k} \left\| \sum_{j=1}^{m+k_m^0} \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \ddot{\mathbf{f}}(\mathbf{X}_j; \tilde{\boldsymbol{\beta}}_{m,k}) (\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0) \right\|_2 \\ &\quad + \frac{\|\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0\|_2^4}{4(m+k)} \sum_{j=1}^{m+k_m^0} \|\ddot{\mathbf{f}}(\mathbf{X}_j; \tilde{\boldsymbol{\beta}}_{m,k})\|_2^2. \end{aligned}$$

By the Bienaymé-Tchebychev inequality we have that $(m+k)^{-1} \|\sum_{j=1}^{m+k_m^0} \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \ddot{\mathbf{f}}(\mathbf{X}_j; \tilde{\boldsymbol{\beta}}_{m,k})\|_2 \leq (m+k)^{-1} \sum_{j=1}^{m+k_m^0} \|\dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0)\|_2 \|\ddot{\mathbf{f}}(\mathbf{X}_j; \tilde{\boldsymbol{\beta}}_{m,k})\|_2 \leq ((m+k)^{-1} \sum_{j=1}^{m+k_m^0} \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0))^{1/2} ((m+k)^{-1} \sum_{j=1}^{m+k_m^0} \|\ddot{\mathbf{f}}(\mathbf{X}_j; \tilde{\boldsymbol{\beta}}_{m,k})\|_2^2)^{1/2}$.

Since Θ is compact, together the assumptions (A2), (A4), we obtain that $\text{Var}_{m,k}^*[\tilde{Z}_{m,k}(i)] \leq C$. As in the linear case, using $\sum_{l=1}^{T_m} g^{-2}(m, l, \gamma) \leq C$, we obtain the inequality (39) for \tilde{I}_4 .

For \tilde{I}_7 . We can prove in a similar way as in the Lemma 7.3, that $\mathbf{E}_{m,k}^*[g^{-1}(m, \text{Thebychevl}, \gamma) \tilde{I}_7] = o_{\mathbf{P}}(1)$. Similar arguments as for \tilde{I}_6 of the Lemma 7.3, unlike that $\text{Var}^*[\tilde{I}_7] = O_{\mathbf{P}}(1)$ uniformly in k and probability 1, we obtain the relation (39) for $j = 7$, by the Bienaymé-Tchebychev inequality, Proposition 4.1, and inequality (12).

For \tilde{I}_5 and \tilde{I}_8 . We consider the random variable defined by

$$\tilde{\tilde{Z}}_{m,k}(i) \equiv \mathbb{1}_{\mathcal{U}_{m,k}(i) > m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k}) \right] - \frac{1}{m+k} \sum_{j=m+k_m^0+1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})],$$

and the results are proved by a similar way using assumption (A5) on the place of (A4). \blacksquare

We recall the notations $\hat{\sigma}_{m,k}^{(*)2} = (m - q)^{-1} \sum_{i=1}^m [\varepsilon_{m,k}^*(i) - m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{m,k}^*(j) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)]^2$ and $\hat{\sigma}_{m,k}^2 = (m + k)^{-1} \sum_{i=1}^{m+k} (\varepsilon_i - \bar{\varepsilon}_{m+k})^2$. We will prove that, under hypothesis H_0 , $\hat{\sigma}_{m,k}^2$ and $\hat{\sigma}_{m,k}^{(*)2}$ are two uniformly consistent estimators for the variance σ^2 of the errors ε^2 . Under hypothesis H_1 , this two statistics are significantly different.

Lemma 7.5 *Suppose that the assumptions (A1)-(A4) hold.*

a) *Under the hypothesis H_0 , we have, in probability,*

$$\sup_{1 \leq k \leq T_m} \mathbf{P}_{m,k}^* \left(\left| \frac{\hat{\sigma}_{m,k}}{\hat{\sigma}_{m,k}^{(*)}} - 1 \right| \geq \epsilon \right) \xrightarrow{m \rightarrow \infty} 0.$$

b) *If furthermore the assumption (A5) holds, under the hypothesis H_1 , for all $\epsilon > 0$, there exists a constant $M > 0$ such that*

$$\sup_{1 \leq k \leq T_m} \mathbf{P}_{m,k}^* \left(\left| \frac{\hat{\sigma}_{m,k}}{\hat{\sigma}_{m,k}^{(*)}} - 1 \right| \geq M \right) \leq \epsilon + o_{\mathbf{P}}(1).$$

Proof of Lemma 7.5

We have the decomposition, for each i of 1 to m : $\varepsilon_{m,k}^*(i) - (m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{m,k}^*(j)) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \equiv \sum_{j=1}^8 J_j(m, k, i)$, where: $J_1(m, k, i) = \varepsilon_{m,k}(i)$, $J_2(m, k, i) = - \left(m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{m,k}(j) \right) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)$,

$$J_3(m, k, i) = \mathbb{1}_{k \leq k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(i) < m+k_m^0} \left[f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k}) - \frac{1}{m+k} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})] \right],$$

$$J_4(m, k, i) = \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(i) < m+k_m^0} [f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k})] - \mathbb{1}_{k > k_m^0} \frac{1}{m+k} \sum_{j=1}^{m+k_m^0} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})],$$

$$J_5(m, k, i) = \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(i) > m+k_m^0} [f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(i)}; \hat{\boldsymbol{\beta}}_{m+k})] - \mathbb{1}_{k > k_m^0} \frac{1}{m+k} \sum_{j=m+k_m^0+1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})],$$

$$J_6(m, k, i) = - \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbb{1}_{k \leq k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(j) < m+k_m^0} [f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \hat{\boldsymbol{\beta}}_{m+k})] \right) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \\ + \frac{1}{m+k} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})],$$

$$\begin{aligned}
J_7(m, k, i) &= - \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(j) < m+k_m^0} [f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \boldsymbol{\beta}^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \hat{\boldsymbol{\beta}}_{m+k})] \right) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \\
&\quad + \mathbb{1}_{k > k_m^0} \frac{1}{m+k} \sum_{j=1}^{m+k_m^0} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})], \\
J_8(m, k, i) &= - \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbb{1}_{k > k_m^0} \mathbb{1}_{\mathcal{U}_{m,k}(j) > m+k_m^0} [f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_{\mathcal{U}_{m,k}(j)}; \hat{\boldsymbol{\beta}}_{m+k})] \right) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \\
&\quad + \mathbb{1}_{k > k_m^0} \frac{1}{m+k} \sum_{j=m+k_m^0+1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}_m^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})].
\end{aligned}$$

Then

$$\hat{\sigma}_{m,k}^{(*)2} = \frac{1}{m-q} \sum_{i=1}^m \left[\sum_{j=1}^8 J_j(m, k, i) \right]^2. \quad (40)$$

Under H_0 , we have $J_4, J_5, J_7, J_8 = 0$.

Following results hold under the two hypotheses H_0 and H_1 . For J_1 we have that for any $\epsilon > 0$

$$\sup_{1 \leq k} \mathbf{P}_{m,k}^* \left[\left| \frac{1}{m-q} \frac{1}{\hat{\sigma}_{m,k}^2} \sum_{i=1}^m J_1^2(m, k, i) - 1 \right| \geq \epsilon \right] \xrightarrow{m \rightarrow \infty} 0. \quad (41)$$

For J_2 , we have $E_{m,k}^*[J_2(m, k, i)] = -\bar{\varepsilon}_{m+k} m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)$. Then, using the independence of ε_i and of \mathbf{X}_i , assumption (A4), we obtain the convergence in probability, uniformly in k , as $m \rightarrow \infty$,

$$E_{m,k}^* \left[\frac{1}{m} \sum_{i=1}^m J_2(m, k, i) \right] = -\bar{\varepsilon}_{m+k} \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \mathbf{B}_m^{-1} \frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \right) \rightarrow 0 \cdot \mathbf{A}' \mathbf{B}^{-1} \mathbf{A} = 0.$$

We have also the approximation of $E_{m,k}^*[m^{-1} \sum_{i=1}^m J_2^2(m, k, i)]$ by $\mathbf{B}_m^{-1} E_{m,k}^*[m^{-2} \sum_{j=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \cdot \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{\mathcal{U}_{m,k}(j)}^2 + 2m^{-2} \sum_j \sum_{j' \neq j} \dot{\mathbf{f}}^t(\mathbf{X}_j; \boldsymbol{\beta}^0) \dot{\mathbf{f}}(\mathbf{X}_{j'}; \boldsymbol{\beta}^0) \varepsilon_{\mathcal{U}_{m,k}(j)} \varepsilon_{\mathcal{U}_{m,k}(j')}] = \mathbf{B}_m^{-1} [m^{-1} \mathbf{B}_m(m+k)^{-1} \sum_{j=1}^{m+k} \varepsilon_j^2 + 2m^{-2} \sum_j \sum_{j' \neq j} \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \cdot \dot{\mathbf{f}}(\mathbf{X}_{j'}; \boldsymbol{\beta}^0) ((m+k)^{-1} \sum_{a=1}^{m+k} \varepsilon_a)^2]$. But $(m+k)^{-1} \sum_{a=1}^{m+k} \varepsilon_a \xrightarrow{P} 0$ and $(m+k)^{-1} \sum_{a=1}^{m+k} \varepsilon_a^2 \xrightarrow{P} \sigma^2$. Hence, we have uniformly in k

$$E_{m,k}^* [m^{-1} \sum_{i=1}^m J_2^2(m, k, i)] \xrightarrow{P} 0. \quad (42)$$

On the other hand, we can write $J_3(m, k, i) = \mathcal{Z}_{k,m}(i)$, with $\mathcal{Z}_{k,m}(i)$ defined by the relation (38). By the proof of the Lemma 7.3, since $\text{Var}_{m,k}^*[\mathcal{Z}_{k,m}(i)] = (m+k)^{-1}\mathbf{B}(1+o_P(1))$ and the Bienaymé-Tchebychev inequality, we have in probability

$$\sup_{k \geq 1} \mathbf{P}_{m,k}^* \left[\frac{1}{m-q} \sum_{i=1}^m J_3^2(m, k, i) \geq \epsilon \right] \xrightarrow{m \rightarrow \infty} 0. \quad (43)$$

For $J_1 J_2$ we have $m^{-1} \sum_{i=1}^m J_1(m, k, i) J_2(m, k, i) = - \left(m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{\mathcal{U}_{m,k}(j)} \right) \mathbf{B}_m^{-1} \left(m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \varepsilon_{\mathcal{U}_{m,k}(j)} \right)$ and then, as for the calculations from above for J_2 , we have $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_1(m, k, i) J_2(m, k, i)] = o_P(1)$, uniformly in k .

For $J_1 J_3$ we have, using the assumptions (A2)-(A4), $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_1(m, k, i) J_3(m, k, i)] = m^{-1} \sum_{i=1}^m \{(m+k)^{-1} \sum_{j=1}^{m+k} \varepsilon_j [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})] \cdot -\bar{\varepsilon}_{m+k} (m+k)^{-1} \sum_{j=1}^{m+k} \varepsilon_j [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})]\} = o_P(1) - o_P(1) o_P(1) = o_P(1)$. We show similar for the other cases that $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_1(m, k, i) J_l(m, k, i)] \xrightarrow{m \rightarrow \infty} 0$, $l = 3, 4, \dots, 8$.

The conditional expectation $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_2(m, k, i) J_3(m, k, i)]$ is equal to

$$\begin{aligned} & -\frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \frac{1}{m} \frac{1}{m+k} \sum_{j=1}^{m+k} \varepsilon_j [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})] \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \\ & + \bar{\varepsilon}_{m+k} \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \right) \mathbf{B}_m^{-1} \left(\frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \right) \frac{1}{m+k} \sum_{j=1}^{m+k} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})], \end{aligned}$$

which converges to 0, from $m \rightarrow \infty$, uniformly in k , since ε_i is independent of \mathbf{X}_i , $\bar{\varepsilon}_{m+k} \rightarrow 0$, and all terms in \mathbf{X}_i are bounded.

a) Under hypothesis H_0 . For $J_6(m, k, i)$ we have

$$\mathbf{E}_{m,k}^*[J_6(m, k, i)] = -(\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0)^t \left(\frac{1}{m+k} \sum_{a=1}^{m+k} \dot{\mathbf{f}}(\mathbf{X}_a; \boldsymbol{\beta}^0) \right) \left(\frac{1}{m} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \right) \mathbf{B}_m^{-1} \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0).$$

Then, under assumption (A4), $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_6(m, k, i)] = -(\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0)^t \mathbf{A} \cdot \mathbf{A}^t \cdot \mathbf{B}^{-1} \mathbf{A} (1 + o_P(1))$. Similarly, with C a constant, $\mathbf{E}_{m,k}^*[J_6^2(m, k, i)] \leq \|\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0\|_2^2 \cdot \|\mathbf{A} \mathbf{A}^t\|_2^2 \cdot \|\mathbf{B}^{-1}\|_2^2 \cdot \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0)^t \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) (1 + o_P(1))$, thus $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_6^2(m, k, i)] \leq \|\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0\|_2^2 \cdot \|\mathbf{A} \mathbf{A}^t\|_2^2 \cdot \|\mathbf{B}^{-1}\|_2^2 \cdot \|\mathbf{B}\|_2 \cdot (1 + o_P(1))$ and $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_6^4(m, k, i)] = C \|\hat{\boldsymbol{\beta}}_{m+k} - \boldsymbol{\beta}^0\|_2^4 (1 + o_P(1))$. Hence, under H_0 , $\text{Var}_{m,k}^*[m^{-1} \sum_{i=1}^m J_6^2(m, k, i)] = o_P(1)$. By the Bienaymé-Tchebychev inequality, we have in probability:

$$\sup_{k \geq 1} \mathbf{P}_{m,k}^* \left[\frac{1}{m-q} \sum_{i=1}^m J_6^2(m, k, i) \geq \epsilon \right] \xrightarrow{m \rightarrow \infty} 0.$$

The relations (40), (41) and since all other conditional expectations for $\hat{\sigma}_{m,k}^{(*)2}$ expression are negligible, imply the assertion (a).

b) Under H_1 . For $b \in \{4, 5, 6, 7, 8\}$ we will prove that for all $\epsilon > 0$ there exists a $M > 0$ such that

$$\sup_{k \geq 1} \mathbf{P}_{m,k}^* \left[\frac{1}{m-q} \sum_{i=1}^m J_b^2(m, k, i) \geq M \right] \leq \epsilon + o_P(1). \quad (44)$$

In view of the previous calculus for J_6 we have that the relation (44) holds for $b = 6$.

For $J_4(m, k, i)$ we have $J_4(m, k, i) = \tilde{Z}_{k,m}(i)$ and by the proof of the Lemma 7.4 and the Bienaymé-Tchebychev inequality, we have that the relation (44) holds for $b = 4$.

For J_7 , its conditional expectation $\mathbf{E}_{m,k}^*[J_7(m, k, i)]$ is, by Taylor expansions and using assumptions (A2), (A4),

$$\frac{(\boldsymbol{\beta}^0 - \hat{\boldsymbol{\beta}}_{m+k})^t}{m+k} \sum_{a=1}^{m+k_m^0} \left[\dot{\mathbf{f}}(\mathbf{X}_a; \boldsymbol{\beta}^0) + \ddot{\mathbf{f}}(\mathbf{X}_a; \tilde{\boldsymbol{\beta}}) \frac{\boldsymbol{\beta}^0 - \hat{\boldsymbol{\beta}}_{m+k}}{2} \right] \left[1 - \frac{1}{m} \mathbf{B}_m^{-1} \sum_{j=1}^m \dot{\mathbf{f}}(\mathbf{X}_j; \boldsymbol{\beta}^0) \right] \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) (1 + o_P(1)).$$

Similarly $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_7^2(m, k, i)] = C \|\boldsymbol{\beta}^0 - \hat{\boldsymbol{\beta}}_{m+k}\|_2^2 (1 + o_P(1))$, $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_7^4(m, k, i)] = C \|\boldsymbol{\beta}^0 - \hat{\boldsymbol{\beta}}_{m+k}\|_2^4 (1 + o_P(1))$ which imply the relation (44) for J_7 . By Bienaymé-Tchebychev inequality, we obtain (44) for J_5 and J_8 using assumption (A5) on the place of (A4).

Now we consider the product of the terms of different suffix. The products of J_3 with J_4, J_5, J_7, J_8 , of J_4 with J_5, J_6, J_8 and of J_5 with J_6, J_7 are 0. For $J_1 J_4$ we have $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_1(m, k, i) J_4(m, k, i)] = (m+k)^{-1} \sum_{j=1}^{m+k_m^0} [\varepsilon_j - \bar{\varepsilon}_{m+k}] [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})]$. But, for all $\epsilon > 0$ there exists $M > 0$ such that $\mathbf{P} \left[(m+k)^{-1} \sum_{j=1}^{m+k_m^0} |f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})| > M \right] < \epsilon$, from which, together the fact $\mathbf{E}[\varepsilon_j] = 0$, one may deduce that

$\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_1(m, k, i) J_4(m, k, i)] \xrightarrow[m \rightarrow \infty]{P} 0$, uniformly in k . By similar arguments we prove the uniformly convergence to 0 in probability, for all other combinations of J_2 and J_4, \dots, J_8 . The not insignificant terms are $J_4^2, J_7^2, J_5^2, J_8^2$. We consider now $J_4 J_7$, the other cases are similar. Taking into account the fact that $\mathbf{E}_{m,k}^*[f(\mathbf{X}_{u_{m,k}(i)}; \boldsymbol{\beta}) f(\mathbf{X}_{u_{m,k}(j)}; \boldsymbol{\beta})]$ is equal to $\mathbf{E}_{m,k}^*[f(\mathbf{X}_{u_{m,k}(i)}; \boldsymbol{\beta})] \mathbf{E}^*[f(\mathbf{X}_{u_{m,k}(j)}; \boldsymbol{\beta})]$ for $i \neq j$ and to $\mathbf{E}^*[f^2(\mathbf{X}_{u_{m,k}(i)}; \boldsymbol{\beta})]$ for $i = j$, we have that $\mathbf{E}_{m,k}^*[m^{-1} \sum_{i=1}^m J_4(m, k, i) J_7(m, k, i)] =$

$$\frac{1}{m(m+k)} \sum_{j=1}^{m+k_m^0} [f(\mathbf{X}_j; \boldsymbol{\beta}^0) - f(\mathbf{X}_j; \hat{\boldsymbol{\beta}}_{m+k})]^2 \left(\frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) \right) \mathbf{B}_m^{-1} \left(\frac{1}{m} \sum_{i=1}^m \dot{\mathbf{f}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \right) (1 + o_P(1)),$$

which converges to 0 in probability, uniformly in k .

Hence, in conclusion, taking into account the relations (40), (42), (43) and (44),

$$\hat{\sigma}_{m,k}^{2(*)} = \frac{1}{m-q} \left[\sum_{i=1}^m (J_1^2(m, k, i) + J_4^2(m, k, i) + J_5^2(m, k, i) + J_7^2(m, k, i) + J_8^2(m, k, i)) \right] (1 + o_P(1))$$

$$\geq \frac{1}{m} \sum_{i=1}^m J_1^2(m, k, i)(1 + o_P(1))$$

and the assertion (b) follows by (41). ■

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